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Krishna Rao V. Kaza

and

Raymond G. Kvaternik

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National Aeronautics and Space Administration

Langley Research Center
Hampton, Virginia 23665

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Krishna Rao V. Kaza
NASA Lewis Research Center
Cleveland, Ohio 44135
and
The University of Toledo
Toledo, Ohio 43606

and

Raymond G. Kvaternik
NASA Langley Research Center
Hampton, Virginia 23665

SUMMARY

The second-degree nonlinear aeroelastic equations for a flexible, twisted nonuniform rotor blade which is undergoing combined flapwise bending, chordwise bending, torsion, and extension in forward flight are developed using Hamilton's principle. The derivation of the equations has its basis in the geometric nonlinear theory of elasticity and the resulting equations are consistent with the small deformation approximation in which the elongations and shears are negligible compared to unity and the square of the derivative

of the extensional deformation of the elastic axis is negligible compared to the squares of the bending slopes. The implications of the slender beam approximation as applied to the derivation of the second-degree nonlinear equations of motion are discussed and a mathematical ordering scheme which is compatible with the assumption of a slender beam is intoduced. No assumption is made regarding the coincidence of the elastic, mass, and tension axes of the blade, although the elastic and aerodynamic center axes are assumed coincident at the blade quarter chord. The blade aerodynamic loading is obtained from strip theory based on a quasi-steady approximation of two-dimensional, incompressible unsteady airfoil theory. The resulting equations are compared with several of those existing in the literature. These comparisons indicate several discrepancies with the present equations, particularly in the nonlinear terms. The reasons for these discrepancies are explained.

INTRODUCTION

Flap-lag-torsion aeroelastic stability of flexible helicopter rotor blades has been receiving considerable attention in the literature during the last decade. This problem involves both linear and nonlinear coupling among the various degrees of freedom. Current emphasis in the literature (see, for example, references 1 to 10) is being directed at the nonlinear aspects of the problem as arising from the nonlinear theory of elasticity, either directly or indirectly. In general, the nonlinearity of the equations of the theory of elasticity can have both geometrical and physical origin (references 11 and 12). Geometric nonlinearity is associated with the necessity to consider the deformed configuration in writing the equilibrium equations and the need to include nonlinear terms in the strain-displacement relations. Physical nonlinearity is associated with the necessity to consider the relations between the components of stress and strain as nonlinear. In the present development only geometrical nonlinearity is considered. The equations are derived using the level of approximation identified as small deformations I in reference 13. This level of approximation assumes that the elongations and shears are negligible compared to unity and that the square of the derivative of the extensional deformation on the elastic axis is negligible compared to the squares of the bending slopes. The equations of motion consistent with this level of approximation may be derived to any desired degree by retaining the dependent variables to the appropriate degree throughout the development. The present development will be directed to the derivation of the second-degree nonlinear equations of motion in which one formally retains terms through second degree in the dependent variables. Rigorous adherence to this retention scheme leads to an almost insurmountable amount of algebra. To circumvent this problem to some extent, usual practice in the literature dealing with flexible rotor blades is to introduce an ordering scheme early in the development of the equations. Following this practice, an ordering scheme which is consistent with the assumption of a slender beam is imposed early in the development of the dynamic and elastic portions of the present equations. The ordering scheme imposed has both a mathematical and physical basis and is discussed in Appendix A. No ordering scheme is imposed in the development of the generalized aerodynamic forces herein because any ordering scheme which is imposed would depend on the order assigned to the advance ratio, inflow ratio, and collective and cyclic pitch, which in turn depend on the flight condition being addressed. To accommodate any flight regime of interest with the present equations, the aerodynamic forces are left in general second-degree form from which one can obtain the aerodynamic forces to the order appropriate to any case of interest.

The generalized aerodynamic forces are obtained from strip theory based on a quasi-steady approximation of two-dimensional, incompressible, unsteady airfoil theory. The effects of reverse flow and stall are not considered. Consideration of forward flight leads to aerodynamic forces which are periodic. The solution of the resulting equations requires special procedures such as Floquet-Liapunov theory or time history solutions by direct numerical integration. However, in the special case of hover for a rotor having three or more blades, the resulting equations have constant coefficients and can be solved using standard eigenvalue techniques.

It was shown in reference 14 that the existence of some linear aerodynamic coupling terms associated with blade steady-state flapping and lagging in the perturbation equations for a rigid articulated blade was dependent on the order in which the flap and lag rotational transformations were imposed while developing the nonlinear equations of motion. The need for addressing the order in which the component rotations are imposed when developing nonlinear equations of motion is a consequence of the fact that the angles of rotation associated with the flapping and lagging motions must be treated as finite. In this case, the matrices associated with the individual rotations are not commutative. A preliminary study of the role of the assumed transformation sequence in the development of the nonlinear flap-lag equations for a flexible blade was also given in reference 14. The need for addressing the order in which the component rotations are imposed also arises while developing the nonlinear equations for an elastic blade. In this case, the angles of rotation associated with the deformations must be treated as finite and the matrices associated with the individual rotations are not commutative. On the basis of those preliminary considerations, it was shown that aerodynamic coupling terms similar to those found for a rigid blade will also appear in the equations for a flexible blade. In addition to differences in the aerodynamic terms, reference 14 also showed that the nonlinear curvature expressions which are needed in deriving the strain expressions are also dependent on the order in which the rotational transformations are imposed. Reference 13 was directed at more completely examining the effect of the rotational transformation sequence on the nonlinear curvatures. As an extension of the work in reference 13, this report presents an extensive development of the nonlinear aeroelastic equations in the presence of coupled flapwise bending, edgewise bending, torsion, and extension, and then examines the effect of the assumed rotational transformation sequence on the form of the equations. Out of the six possible rotational transformation sequences which may be imposed, only two will be addressed here: flap-lag-pitch and lag-flap-pitch.

The present equations will be compared to several sets of corresponding equations existing in the literature. Several discrepancies with the present results will be identified, particularly in the nonlinear terms. The reasons for these discrepancies will be explained. Furthermore, it will be shown that the particular small deformation approximation considered in the present report is necessary to insure the retention of all the linear and second-degree nonlinear terms in the second-degree nonlinear equations of motion.

SYMBOLS

a	airfoil lift-curve-slope
A	cross-sectional area of blade
A_u, A_v, A_w	generalized aerodynamic forces per unit length in X,Y,Z directions, respectively
A_ϕ	generalized aerodynamic moment per unit length about elastic axis
b	number of blades
$B_V, B_T, B_{\delta W_D}$	boundary terms arising from strain energy, kinetic energy, and material damping, respectively
B_1, B_2	section constants
c	blade chord
C_{d_o}	airfoil profile drag coefficient
$C(k)$	Theodorsen's circulation function
C_T	rotor thrust coefficient, $T/\rho\pi\Omega^2R^4$
C_1, C_2	section constants
d_i $(i = 1,2...6)$	notation used in writing the virtual work associated with material damping in concise form
D_1, D_2	section constants
D	airfoil profile drag per unit length

e	chordwise offset of mass centroid from elastic axis (positive when in front of elastic axis)
e_A	chordwise distance of area centroid of cross section from elastic axis (positive when in front of elastic axis)
E	Young's modulus
E^*	coefficient of internal friction in tension
$\bar{e}_{x_3},\bar{e}_{y_3},\bar{e}_{z_3}$	unit vectors along x_3,y_3,z_3 axes
$\bar{e}_X,\bar{e}_Y,\bar{e}_Z$	unit vectors along XYZ axes
F_{x_3},F_{y_3},F_{z_3}	components of aerodynamic force per unit length in x_3,y_3,z_3 directions
G	shear modulus
G^*	coefficient of internal friction in shear
$\dot{h}$	vertical velocity of two-dimensional section normal to free-stream
I_u,I_v,I_w	generalized inertia forces per unit length in X,Y,Z directions
I_ϕ	generalized inertia moment per unit length about elastic axis
$I_{\eta\eta},I_{\zeta\zeta}$	area moments of inertia about η and ζ axes, respectively
J	torsional section constant
k	reduced frequency

k_A	polar radius of gyration of cross-sectional area about elastic axis
k_i $(i = 1,2...6)$	notation used in writing the variation of the kinetic energy in concise form
k_m	polar radius of gyration of cross-sectional mass about elastic axis $(k_m{}^2 = k_{m_1}{}^2 + k_{m_2}{}^2)$
k_{m_1}, k_{m_2}	mass radii of gyration about η and ζ axes, respectively
L	aerodynamic lift per unit length
M, M_ϕ, M_{x_3}	aerodynamic pitching moment per unit length about the deformed elastic axis
m	mass of blade per unit length
$Q_{D_u}, Q_{D_v}, Q_{D_w}, Q_{D_\phi}$	generalized damping forces
R	length of blade
$\bar{r}_1$	position vector of point after deformation
$\bar{r}_o$	position vector of point before deformation
s_i $(i = 1,2...9)$	notation used in writing the variation of the strain energy in concise form
S_u, S_v, S_w, S_ϕ	generalized elastic forces
T	kinetic energy; also blade tension; also rotor thrust
U_R, U_T, U_P	radial, tangential, and perpendicular components of

	velocity for blade airfoil section
U	resultant of U_T and U_P
U_F	radial foreshortening of elastic axis due to bending
u,v,w	deformations of elastic axis in X,Y, and Z directions, respectively
V	strain energy; also forward flight velocity
$\bar{V}_{XYZ}$	relative velocity of point on elastic axis expressed in XYZ coordinate system
$\bar{V}_{x_3y_3z_3}$	relative velocity of point on elastic axis expressed in $x_3y_3z_3$ coordinate system
$\bar{V}_a$	wind velocity vector
v_i	induced downwash velocity at rotor, positive downward
W	work done by nonconservative forces
W_A	work done by aerodynamic loading
W_D	work done by structural damping
XYZ	coordinate system with origin at hub centerline which rotates with blade such that X-axis lies along the initial or undeformed position of the elastic axis
$X_IY_IZ_I$	inertial axis system with origin at hub centerline and Z_I normal to hub plane

$X_\Omega Y_\Omega Z_\Omega$	hub-fixed axis system rotating about the Z_I axis with angular velocity Ω
xyz	blade-fixed axis system, after deformation, which translates with respect to $x_o y_o z_o$
$x_o y_o z_o$	blade-fixed axis system at arbitrary p[illegible] on elastic axis before deformation
$x_1 y_1 z_1$	coordinates of point (which was at $x_o y_o z_o$ in the undeformed blade) in the deformed blade
$x_3 y_3 z_3$	blade-fixed orthogonal axis system in deformed configuration obtained by rotating xyz; x_3-axis is tangent to the deformed elastic axis
$[T]$	transformation matrix relating the angular orientation of the deformed and undeformed blade
$[\varepsilon_{ij}]$	Green's strain tensor
α	airfoil section angle of attack, $\alpha = \tan^{-1} U_P/U_T$
α_s	shaft angle
β_{pc}	angle of built-in coning (precone angle)
β, ζ, θ	Eulerian-type rotation angles between xyz and $x_3 y_3 z_3$
$\gamma_{xx}, \gamma_{x\eta}, \gamma_{x\zeta}$	engineering strain components
$\delta(\ )$	variation of ( )
ε	small parameter of the order of the bending slopes; also

	airfoil section pitch angle with respect to free-stream velocity
$\varepsilon_{xx},\varepsilon_{x\eta},\varepsilon_{x\zeta}$	tensor strain components
η	sectional coordinate along major principal axis for a given point on the elastic axis
$\hat{\eta}$	$\eta - \lambda_\zeta$
ζ	sectional coordinate normal to η axis at elastic axis
$\hat{\zeta}$	$\zeta + \lambda_\eta$
θ_c	collective pitch
θ_{1c},θ_{1s}	cyclic pitch components
θ_{kc}	pitch angle due to kinematic coupling
θ_{pt}	built-in twist (pretwist), positive when leading edge is upward
$\lambda(\eta,\zeta)$	warping function
$\lambda_\eta,\lambda_\zeta$	derivatives of λ with respect to η and ζ, respectively
λ	inflow ratio, $\lambda\Omega R = V\sin\alpha_s - v_i$, positive upward
μ	advance ratio, $\mu\Omega R = V\cos\alpha_s$
ρ	mass density of blade; also mass density of air
θ	total geometric pitch angle

$\sigma_{xx},\sigma_{x\eta},\sigma_{x\zeta}$ $\tau_{xx},\tau_{x\eta},\tau_{x\zeta}$	engineering stresses
ϕ	angle of twisting deformation about elastic axis, positive when leading edge is upward
ψ	blade azimuth angle measured from downwind position in direction of rotation
ω_{x_3}	torsional curvature
$\bar{\omega}$	angular velocity of XYZ coordinate system
Ω	rotational speed of rotor
$(\;)_C$	circulatory aerodynamic term
$(\;)_{NC}$	noncirculatory aerodynamic term
$[\;]^T$	denotes transpose of matrix
$(\dot{\;})$	time derivative $\frac{\partial}{\partial t}$
$(\;)'$	space derivative $\frac{\partial}{\partial x}$

MATHEMATICAL MODEL AND COORDINATE SYSTEMS

The mathematical model chosen to represent the rotor blade in the present development consists of a straight, slender, variably twisted, nonuniform beam which can undergo combined flapwise bending, chordwise bending, torsion, and extension. The elastic axis, the mass axis, and the tension axis (area-centroid axis) are taken to be noncoincident; the elastic axis and the feathering axis are assumed coincident with the quarter-chord of the blade. The elastic axis is inclined to the plane of rotation at a small angle in order to accommodate any built-in coning (precone). The model is valid for blade-root conditions at the shaft centerline which represent either clamped (hingeless) or pinned (articulated) configurations. In the latter case, the precone angle would be set to zero. Based on a quasi-steady approximation of two-dimensional, unsteady airfoil theory, a distributed aerodynamic loading in the flapwise and lagwise directions and a distributed aerodynamic torque about the elastic axis is assumed to be acting on the blade.

Several orthogonal coordinate systems will be employed in the derivation of the equations of motion; those which are common to both the dynamic and aerodynamic aspects of the derivation are shown in figures 1 to 4. The axis systems associated with the blade in its undeformed configuration are given in figures 1 and 2. The axis system $X_I Y_I Z_I$ (figure 1) is fixed in an inertial frame with origin at the centerline of the hub, and the Z_I axis is normal to the plane of the hub. The axis system $X_\Omega Y_\Omega Z_\Omega$ is obtained by rotating about the positive Z_I axis by the angle $\psi = \Omega t$, where Ω is the constant angular velocity of the rotor blade. The third axis system shown in figure 1, XYZ, is obtained by rotating $X_\Omega Y_\Omega Z_\Omega$ about the negative Y_Ω axis by an amount β_{pc}, the angle of built-in coning. All deformations of the blade are referenced to the XYZ system. The X axis is taken to be aligned along the elastic axis of the undeformed blade. As mentioned above, it is assumed that the elastic axis, the feathering axis, and the quarter-chord of the blade are coincident. The geometry of a cross section of the blade at an arbitrary spanwise station along the X axis before deformation is shown in figure 2. The point of the cross section through which the elastic axis passes is given by the intersection of the Y and Z axes. The η and ζ axes with origin at the elastic axis are principal axes of the cross section and are inclined to the Y and Z axes by an amount equal to the total geometric pitch angle, θ. The geometric pitch angle is given by

$$\theta = \theta_{pt} + \theta_{kc} + \theta_c - \theta_{1c}\cos\psi - \theta_{1s}\sin\psi \tag{1}$$

where θ_{pt} is the built-in twist angle (pretwist), θ_{kc} is the pitch angle due to kinematic coupling, θ_c is the collective pitch angle, and θ_{1c} and θ_{1s} are the first harmonic cyclic pitch components. The pretwist (θ_{pt}) is a function of the running coordinate x while the collective and cyclic pitch components of the control input $(\theta_c, \theta_{1c}, \theta_{1s})$ are independent of both x and t. The pitch angle due to kinematic coupling (θ_{kc}) is to be included in equation 1 only if the elastic blade has hinges at the root. In this case, θ_{kc} is dependent on the rigid-body flapping and lagging motions. The cross section is assumed to be symmetric with respect to the η axis. During deformation, the η and ζ axes are assumed to move with the cross section.

The generalized coordinates defining the configuration of the deformed blade are shown in figures 3 and 4. The situation depicted in figure 3 is appropriate to a rotational transformation sequence which is flap, followed by lag, followed by pitch (flap-lag-pitch sequence) while that shown in figure 4 is appropriate to a rotational transformation sequence which is lag, followed by flap, followed by pitch (lag-flap-pitch sequence). When the blade deforms, the elastic axis at an arbitrary section deforms an amount u in the X direction, v in the Y direction, and w in the Z direction and the section rotates about the principal axes due to bending in addition to twisting an amount ϕ about the elastic axis. Let $x_o y_o z_o$ be axes fixed to the blade at an arbitrary point on the elastic axis of the blade so that before deformation $x_o y_o z_o$ are parallel to XYZ, respectively. The deformations u, v, w, and ϕ both displace $x_o y_o z_o$ to xyz and rotate xyz to $x_3 y_3 z_3$ where the x_3 axis is tangent to the deformed elastic axis. The rotation of the triad xyz to its final position denoted by $x_3 y_3 z_3$ may be effected in several ways depending on the sequence in which the analyst chooses to impose the individual rotations. Two rotational transformation sequences are considered here. Detailed considerations related to these transformations are contained in reference 13.

Some comments regarding the geometric pitch angle are in order. The built-in twist as well as the control inputs and kinematic coupling are present in the blade even before deformation, as shown in figure 2. Then, when imposing a rotational sequence between the xyz and $x_3 y_3 z_3$ axis systems, the rotation θ should be imposed first. However, common practice in the rotor blade literature is to combine pretwist with elastic torsion. For mathematical convenience herein, the control inputs and kinematic coupling will be included with elastic torsion in the same manner as the pretwist.

HAMILTON'S PRINCIPLE

The equations of motion are derived using the extended Hamilton's principle (reference 15) in the form

$$\int_{t_0}^{t_1} (\delta T - \delta V + \delta W)dt = 0 \qquad (2)$$

where

$$\delta W = \delta W_D + \delta W_A \qquad (3)$$

In equation 2, T is the kinetic energy, V is the strain energy, and W is the work done by all the nonconservative forces. For subsequent convenience, the nonconservative work is divided into two parts as indicated in equation 3: the first part, δW_D, due to structural damping and the second part, δW_A, due to the aerodynamic loading. In the following sections explicit expressions for δT, δV, and δW in terms of the dependent variables u, v, w, and ϕ and the blade sectional properties will be developed for two of the six possible rotational transformation sequences which may be imposed in arriving at a relationship between the blade-fixed coordinates of the deformed and undeformed blade. In this development, the geometric nonlinear theory of elasticity, in particular the level of approximation in this theory designated as small deformations I in reference 13, will be employed.

STRAIN ENERGY

The expression for the strain energy of the blade in terms of stresses and engineering strains is

$$V = \frac{1}{2}\int_0^R \iint_A (\sigma_{xx}\gamma_{xx} + \sigma_{x\eta}\gamma_{x\eta} + \sigma_{x\zeta}\gamma_{x\zeta})d\eta\, d\zeta\, dx \tag{4}$$

where, using Hooke's law,

$$\begin{aligned}\sigma_{xx} &= E\,\gamma_{xx}\\ \sigma_{x\eta} &= G\,\gamma_{x\eta}\\ \sigma_{x\zeta} &= G\,\gamma_{x\zeta}\end{aligned} \tag{5}$$

Assuming small strains, the engineering strains are related to the components of the strain tensor according to

$$\gamma_{xx} = \varepsilon_{xx}$$

$$\gamma_{x\eta} = 2\varepsilon_{x\eta}$$

$$\gamma_{x\zeta} = 2\varepsilon_{x\zeta} \tag{6}$$

Several different definitions of strain may be found in the literature (see, for example, reference 16). Adopting a Lagrangian description for the strain (as customary in solid mechanics) wherein measurements are with respect to the initial or undeformed configuration, the appropriate strain tensor is Green's strain tensor $[\varepsilon_{ij}]$ the components of which (reference 11) can be written in the form

$$d\bar{r}_1 \cdot d\bar{r}_1 - d\bar{r}_o \cdot d\bar{r}_o = 2\lfloor dx \; d\eta \; d\zeta \rfloor [\varepsilon_{ij}] \begin{Bmatrix} dx \\ d\eta \\ d\zeta \end{Bmatrix} \tag{7}$$

The quantities $d\bar{r}_o$ and $d\bar{r}_1$ in equation 7 are differentials of the position vectors to an arbitrary point in the blade cross section in the undeformed and deformed configurations, respectively. The scalar quantities $d\bar{r}_o \cdot d\bar{r}_o$ and $d\bar{r}_1 \cdot d\bar{r}_1$ are then the squares of a differential line element before and after deformation, respectively, where dx $d\eta$ $d\zeta$ are increments along the undeformed elastic axis and two cross-sectional axes, respectively. This implies that the strain considered is that along a pretwisted fiber.

The position vector of a generic point in the cross section of the undeformed blade is given by

$$\bar{r}_o = \begin{Bmatrix} x_o \\ y_o \\ z_o \end{Bmatrix} = \begin{Bmatrix} x - \lambda\,\theta'_{pt} \\ \eta\cos\theta - \zeta\sin\theta \\ \eta\sin\theta + \zeta\cos\theta \end{Bmatrix} \tag{8}$$

where the x axis is aligned along the undeformed elastic axis. The

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corresponding point in the deformed blade is given by the sum of the displacement of the elastic axis due to deformation and the position of the point relative to the elastic axis and can be written as

$$\bar{r}_1 = \begin{Bmatrix} x_1 \\ y_1 \\ z_1 \end{Bmatrix} = \begin{Bmatrix} x + u - U_F \\ v \\ w \end{Bmatrix} + [T]^T \begin{Bmatrix} -\lambda\omega_{x_3} \\ \eta \\ \zeta \end{Bmatrix} \tag{9}$$

where U_F, the axial displacement associated with the foreshortening of the elastic axis due to bending, is given by

$$U_F = \tfrac{1}{2} \int_0^x (v'^2 + w'^2)dx \tag{10}$$

The transformation matrix $[T]$ relates the deformed blade axis system $x_3y_3z_3$ with the undeformed blade axis system xyz. The warping function λ is a function of the cross-sectional coordinates η and ζ and is obtained by solving Laplace's equation for the cross section of the blade (reference 17). The quantity ω_{x_3} is the torsional curvature about the deformed elastic axis which has x_3 as its tangent. The quantities U_F, $[T]$, and ω_{x_3} are functions of the dependent variables u, v, w, and ϕ which are in turn functions of x and t. For convenience of notation the functional dependence of the quantities on x, t, η, or ζ will not be indicated.

The elements of the transformation matrix $[T]$ in equation 9 depend on the order in which the sequential rotational transformations from the xyz system to the $x_3y_3z_3$ system are imposed by the analyst, as stated earlier. The particular strains ε_{xx}, $\varepsilon_{x\eta}$, and $\varepsilon_{x\zeta}$ and the associated strain energy will now be developed for each of the two transformation sequences considered herein.

Flap-Lag-Pitch Transformation Sequence

For this particular sequence, the rotational transformation matrix [T] which relates $x_3y_3z_3$ to xyz is obtained from reference 13 by replacing θ_{pt} by θ from equation 1. To second degree, the result is given by

$$[T_{FLP}] = \begin{bmatrix} 1 - \frac{1}{2}\left(v'^2 + w'^2\right) & v' & w' \\ -v'\left(\cos\theta - \phi\sin\theta\right) & \left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta & \left(\phi - v'w'\right)\cos\theta \\ -w'\left(\sin\theta + \phi\cos\theta\right) & -\phi\sin\theta & +\left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta \\ v'\left(\sin\theta + \phi\cos\theta\right) & -\left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta & \left(v'w' - \phi\right)\sin\theta \\ -w'\left(\cos\theta - \phi\sin\theta\right) & -\phi\cos\theta & +\left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta \end{bmatrix} \tag{11}$$

From the same reference, the appropriate expression for the torsional curvature ω_{x_3} is given by

$$\omega_{x_3} = \theta'_{pt} + \phi' - v'w'' \tag{12}$$

As already remarked, reference 13 considered pretwist combined with elastic twist for convenience following earlier practice in the rotor blade literature. The same expedient is employed in the present work. As a consequence of employing this simplification, the position vector of a point before deformation as obtained from equation 9 by setting u, v, w, and ϕ to zero yields the term $-\lambda\theta'_{pt}$ in the x_o component (see equation 8). This implies that axial deformation due to warping exists in the initial configuration before any deformations are imposed. Such a situation would exist if an untwisted blade is twisted and then "frozen" to arrive at the pretwisted configuration. It should also be noted that reference 13 did not explicitly consider foreshortening in the development given therein. However, even if foreshortening is considered explicitly the rotational transformation matrix

[T] remains unchanged for the level of approximation considered in this report.

Substituting equations 11 and 12 into equation 9, the components of the final position vector to second degree are given by

$$x_1 = x + u - U_F - \lambda\omega_{x_3} - (v' + w'\phi)(\eta \cos\theta - \zeta \sin\theta)$$

$$- (w' - v'\phi)(\eta \sin\theta + \zeta \cos\theta)$$

$$y_1 = v - \lambda\omega_{x_3} v' + (1 - \frac{v'^2}{2} - \frac{\phi^2}{2})(\eta \cos\theta - \zeta \sin\theta)$$

$$- \phi\ (\eta \sin\theta + \zeta \cos\theta)$$

$$z_1 = w - \lambda\omega_{x_3} w' + (1 - \frac{w'^2}{2} - \frac{\phi^2}{2})(\eta \sin\theta + \zeta \cos\theta)$$

$$+ (\phi - v'w')(\eta \cos\theta - \zeta \sin\theta) \tag{13}$$

From equations 8 and 13 the differentials of the position vectors before and after deformation, to second degree, are given by

$$dx_o = dx\ (1 - \lambda\theta''_{pt}) - \lambda_\eta \theta'_{pt}\ d\eta - \lambda_\zeta \theta'_{pt}\ d\zeta$$

$$dy_o = -z_o\ \theta'_{pt}\ dx + \cos\theta\ d\eta - \sin\theta\ d\zeta$$

$$dz_o = y_o\,\theta'_{pt}\,dx + \sin\theta\,d\eta + \cos\theta\,d\zeta \tag{14}$$

and

$$\begin{aligned}
dx_1 = \Big[& 1 + u' - U'_F - \lambda\omega'_{x_3} - (v'' + w''\phi + w'\phi')(\eta\cos\theta - \zeta\sin\theta) \\
& + (v' + w'\phi)(\eta\sin\theta + \zeta\cos\theta)\theta'_{pt} \\
& - (w'' - v''\phi - v'\phi')(\eta\sin\theta + \zeta\cos\theta) \\
& - (w' - v'\phi)(\eta\cos\theta - \zeta\sin\theta)\theta'_{pt}\Big]dx \\
& + \Big[-\lambda_\eta\omega_{x_3} - (v' + w'\phi)\cos\theta - (w' - v'\phi)\sin\theta\Big]d\eta \\
& + \Big[-\lambda_\zeta\omega_{x_3} - (w' - v'\phi)\cos\theta + (v' + w'\phi)\sin\theta\Big]d\zeta
\end{aligned} \tag{15a}$$

$$\begin{aligned}
dy_1 = \Big[& v' - \lambda(\omega_{x_3}v'' + v'\omega'_{x_3}) - (v''v' + \phi'\phi)(\eta\cos\theta - \zeta\sin\theta) \\
& - \left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)(\eta\sin\theta + \zeta\cos\theta)\theta'_{pt} \\
& - \phi'(\eta\sin\theta + \zeta\cos\theta) - \phi\theta'_{pt}\,(\eta\cos\theta - \zeta\sin\theta)\Big]dx \\
& + \Big[\left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta - \phi\sin\theta - \lambda_\eta\omega_{x_3}v'\Big]d\eta \\
& - \Big[\left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta + \phi\cos\theta + \lambda_\zeta\omega_{x_3}v'\Big]d\zeta
\end{aligned} \tag{15b}$$

$$dz_1 = \Big[w' - \lambda(\omega_{x_3} w'' + w'\omega'_{x_3}) - (w'w'' + \phi\phi')(\eta \sin\theta + \zeta\cos\theta)$$

$$+ (1 - \frac{w'^2}{2} - \frac{\phi^2}{2})(\eta\cos\theta - \zeta\sin\theta)\theta'_{pt}$$

$$+ (\phi' - v''w' - v'w'')(\eta\cos\theta - \zeta\sin\theta)$$

$$- (\phi - v'w')(\eta\sin\theta + \zeta\cos\theta)\theta'_{pt} \Big] dx$$

$$+ \Big[(1 - \frac{w'^2}{2} - \frac{\phi^2}{2})\sin\theta + (\phi - v'w')\cos\theta - \lambda_\eta \omega_{x_3} w' \Big] d\eta$$

$$+ \Big[(1 - \frac{w'^2}{2} - \frac{\phi^2}{2})\cos\theta - (\phi - v'w')\sin\theta - \lambda_\zeta \omega_{x_3} w' \Big] d\zeta \qquad (15c)$$

Substituting equations 14 and 15 into equation 7, performing the indicated operations, and collecting terms, the second-degree expressions for the three strain components of interest become

$$\gamma_{xx} = \varepsilon_{xx} = u' - \lambda\phi'' + (\eta^2 + \zeta^2)(\phi'\ \theta'_{pt} + \frac{\phi'^2}{2})$$

$$- (v'' + w''\phi)(\eta\cos\theta - \zeta\sin\theta)$$

$$- (w'' - v''\phi)(\eta\sin\theta + \zeta\cos\theta) \qquad (16a)$$

$$\gamma_{x\eta} = 2\varepsilon_{x\eta} = -\hat{\zeta}\phi' + \hat{\zeta}v'w'' \qquad (16b)$$

$$\gamma_{x\zeta} = 2\varepsilon_{x\zeta} = \hat{\eta}\phi' - \hat{\eta}v'w'' \tag{16c}$$

where

$$\hat{\eta} = \eta - \lambda_\zeta$$

$$\hat{\zeta} = \zeta + \lambda_\eta \tag{17}$$

It should be pointed out that in arriving at the expressions given in equations 16 above several terms have been discarded based either on considerations related to the small deformations I level of approximation, as discussed in reference 13, or on considerations related to the approximations which can be made because of the assumed slenderness of the blade, as discussed in reference 18 and in Appendix A. Retention of higher order terms in the expressions for the strain components is not at all a problem. However, these higher order terms in the strains lead to higher order terms in the final equations of motion. Thus, discarding these higher order terms at the strains level using the considerations of Appendix A simplifies the subsequent algebraic manipulations.

Taking the first variation of V as given in equation 4 and using equation 5, yields

$$\delta V = \int_0^R E \iint_A \gamma_{xx}\delta\gamma_{xx}\, d\eta\, d\zeta\, dx + \int_0^R G \iint_A (\gamma_{x\eta}\delta\gamma_{x\eta} + \gamma_{x\zeta}\delta\gamma_{x\zeta}) d\eta\, d\zeta\, dx \tag{18}$$

where the engineering strains are related to the tensor strain as indicated in equation 6. Using equations 16 in equation 18, taking the indicated variations, and integrating over the cross section leads to

$$\delta V = \int_0^R \Big[s_1 \delta u' + s_2 \delta\phi'' + s_3 \delta\phi' + s_4 \delta v'' + s_5 \delta w'' + s_6 \delta\phi$$

$$+ s_7 \delta\phi' + s_8 \delta w'' + s_9 \delta v' \Big] dx \tag{19}$$

where

$$s_1 = EA \Big[u' + k_A{}^2 (\phi' \ \theta'_{pt} + \tfrac{1}{2}\phi'^2) - e_A (v'' + \phi w'') \cos\theta + e_A (\phi v'' - w'') \sin\theta \Big]$$

$$s_2 = EC_1 \phi'' + EC_2 \Big[w'' \cos\theta - v'' \sin\theta - \phi(w'' \sin\theta + v'' \cos\theta) \Big]$$

$$s_3 = EAk_A{}^2 \ u' (\theta'_{pt} + \phi') + EB_1 \Big[\theta'_{pt} \phi'^2 + \theta'_{pt} (\phi' \theta'_{pt} + \tfrac{1}{2}\phi'^2) \Big] + EB_2 \Big[\theta'_{pt} \cdot$$

$$(\phi v'' - w'') \sin\theta - \phi \ v'' \cos\theta - \phi' w'' \sin\theta - \theta'_{pt} (v'' + \phi w'') \cos\theta \Big]$$

$$s_4 = EA \ e_A \ u' (\phi \sin\theta - \cos\theta) - EB_2 \ \phi' \theta'_{pt} \cos\theta - EC_2 \ \phi'' \sin\theta$$

$$+ v'' \Big[EI_{\eta\eta} (\sin^2\theta + \phi \sin 2\theta) + EI_{\zeta\zeta} (\cos^2\theta - \phi \sin 2\theta) \Big]$$

$$+ w'' \Big[(EI_{\zeta\zeta} - EI_{\eta\eta}) (\sin\theta \cos\theta + \phi \cos 2\theta) \Big]$$

$$s_5 = -EAe_A u'(\phi \cos\theta + \sin\theta) + EC_2\phi'' \cos\theta - EB_2\phi'\theta'_{pt} \sin\theta$$

$$+ v''\left[(EI_{\zeta\zeta} - EI_{\eta\eta}) \sin\theta \cos\theta + \phi(EI_{\zeta\zeta} - EI_{\eta\eta}) \cos 2\theta\right]$$

$$+ w''\left[EI_{\eta\eta} \cos^2\theta + EI_{\zeta\zeta} \sin^2\theta + \phi(EI_{\zeta\zeta} - EI_{\eta\eta}) \sin 2\theta\right]$$

$$s_6 = EAe_A u'(v'' \sin\theta - w'' \cos\theta) + EB_2(v''\,\theta'_{pt}\,\phi' \sin\theta - w''\,\theta'_{pt}\phi' \cos\theta)$$

$$+ v''w''\ (EI_{\zeta\zeta} - EI_{\eta\eta}) \cos 2\theta + w''^2(EI_{\zeta\zeta} - EI_{\eta\eta}) \sin\theta \cos\theta$$

$$+ v''^2(EI_{\eta\eta} - EI_{\zeta\zeta}) \sin\theta \cos\theta - EC_2\phi''(v'' \cos\theta + w'' \sin\theta)$$

$$s_7 = GJ\phi' - v'w''(D_1 + D_2)G$$

$$s_8 = -\ \phi'v'(D_1 + D_2)G$$

$$s_9 = -\ \phi'w''(D_1 + D_2)G \qquad (20)$$

The sectional properties appearing in equations 20 are defined as follows:

$$A = \iint d\eta \, d\zeta \qquad A e_A = \iint \eta \, d\eta \, d\zeta$$

$$I_{\eta\eta} = \iint \zeta^2 \, d\eta \, d\zeta \qquad I_{\zeta\zeta} = \iint \eta^2 \, d\eta \, d\zeta$$

$$A k_A^{\,2} = \iint (\eta^2 + \zeta^2) \, d\eta \, d\zeta \qquad J = \iint (\hat{\eta}^2 + \hat{\zeta}^2) \, d\eta \, d\zeta$$

$$B_1 = \iint (\eta^2 + \zeta^2)^2 \, d\eta \, d\zeta \qquad B_2 = \iint \eta \, (\eta^2 + \zeta^2) \, d\eta \, d\zeta$$

$$C_1 = \iint \lambda^2 \, d\eta \, d\zeta \qquad C_2 = \iint \zeta \lambda \, d\eta \, d\zeta$$

$$D_1 = \iint \eta \hat{\eta} \, d\eta \, d\zeta \qquad D_2 = \iint \zeta \hat{\zeta} \, d\eta \, d\zeta \tag{21}$$

Since the warping function $\lambda(\eta, \zeta)$ is typically antisymmetric in η and ζ and the cross section is assumed symmetrical about the η axis, the following integrals are zero:

$$\iint \lambda \, d\eta \, d\zeta = 0 \qquad \iint \lambda (\eta^2 + \zeta^2) \, d\eta \, d\zeta = 0$$

$$\iint \zeta \, d\eta \, d\zeta = 0 \qquad \iint \eta \zeta \, d\eta \, d\zeta = 0$$

$$\iint \zeta (\eta^2 + \zeta^2) \, d\eta \, d\zeta = 0 \qquad \iint \eta \lambda \, d\eta \, d\zeta = 0 \tag{22}$$

It should be observed that D_1 will be identically zero if $\lambda = \eta\zeta$. Although this is not strictly true, for thickness/chord ratios typical of helicopter rotor blades the error in assuming that D_1 is zero is small.

Thus, in the following it will be assumed that $D_1 = 0$. Then the last three expressions given in equation 20 simplify to

$$s_7 = GJ\phi' - v'w''GD_2$$

$$s_8 = -\phi'v'GD_2$$

$$s_9 = -\phi'w''GD_2 \tag{23}$$

Integrating equation 19 by parts, the resulting expression can be put into the form

$$\delta V = \int_0^R (S_u\delta_u + S_v\delta_v + S_w\delta_w + S_\phi\delta_\phi)dx + B_V \tag{24}$$

where the generalized elastic forces S_u, S_v, S_w, and S_ϕ, to second degree, are given by

$$S_u = -s_1'$$

$$S_v = s_4'' - s_9'$$

$$S_w = s_5'' + s_8''$$

$$S_\phi = s_2'' - s_3' + s_6 - s_7' \tag{25}$$

and the boundary terms B_V by

$$B_V = s_1 \delta u \Big|_0^R + (s_9 - s_4') \, \delta v \Big|_0^R + s_4 \delta v' \Big|_0^R - (s_5' + s_8') \, \delta w \Big|_0^R$$

$$+ (s_5 + s_8) \, \delta w' \Big|_0^R + (s_7 + s_3 - s_2') \, \delta\phi \Big|_0^R + s_2 \delta\phi' \Big|_0^R \tag{26}$$

Lag-Flap-Pitch Transformation Sequence

For this sequence, the matrix which relates the deformed blade coordinates $x_3y_3z_3$ to the undeformed blade coordinates xyz is obtained from reference 13 by replacing θ_{pt} by θ from equation 1 and to second degree is given by

$$[T_{LFP}] = \begin{bmatrix} 1 - \frac{1}{2}\left(v'^2 + w'^2\right) & v' & w' \\ \begin{matrix} -v'\left(\cos\theta - \phi\sin\theta\right) \\ -w'\left(\sin\theta + \phi\cos\theta\right) \end{matrix} & \begin{matrix} -\left(\phi + v'w'\right)\sin\theta \\ +\left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta \end{matrix} & \begin{matrix} \left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta \\ +\phi\cos\theta \end{matrix} \\ \begin{matrix} v'\left(\sin\theta + \phi\cos\theta\right) \\ -w'\left(\cos\theta - \phi\sin\theta\right) \end{matrix} & \begin{matrix} -\left(\phi + v'w'\right)\cos\theta \\ -\left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta \end{matrix} & \begin{matrix} \left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta \\ -\phi\sin\theta \end{matrix} \end{bmatrix} \tag{27}$$

From reference 13, the torsional curvature for this case is given by

$$\omega_{x_3} = \theta_{pt}' + \phi' + v''w' \tag{28}$$

Substituting equations 27 and 28 into equation 9, the components of the final position vector to second degree are given by

$$x_1 = x + u - U_F - \lambda\omega_{x_3} - (v' + w'\ \phi)(\eta \cos\theta - \zeta \sin\theta)$$

$$- (w' - v'\ \phi)(\eta \sin\theta + \zeta \cos\theta)$$

$$y_1 = v - \lambda\omega_{x_3} v' + \left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)(\eta \cos\theta - \zeta \sin\theta)$$

$$- (\phi + v'w')(\eta \sin\theta + \zeta \cos\theta)$$

$$z_1 = w - \lambda\omega_{x_3} w' + \left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)(\eta \sin\theta + \zeta \cos\theta)$$

$$+ \phi\ (\eta \cos\theta - \zeta \sin\theta) \tag{29}$$

The position vector of a point before deformation is again given by equation 8. Taking the differentials of the position vectors before and after deformation, substituting the results into equation 7, and collecting terms, the second-degree expressions for the three strain components of interest become

$$\gamma_{xx} = \varepsilon_{xx} = u' - \lambda\phi'' + (\eta^2 + \zeta^2)(\phi'\theta'_{pt} + \frac{1}{2}\phi'^2)$$

$$- (v'' + w''\ \phi)(\eta \cos\theta - \zeta \sin\theta)$$

$$- (w'' - v''\ \phi)(\eta \sin\theta + \zeta \cos\theta) \tag{30a}$$

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$$\gamma_{x\eta} = 2\varepsilon_{x\eta} = - \hat{\zeta}\ \phi' - \hat{\zeta}\ v''w' \tag{30b}$$

$$\gamma_{x\zeta} = 2\varepsilon_{x\zeta} = \hat{\eta}\ \phi' + \hat{\eta}\ v''w' \tag{30c}$$

As before, several higher order terms have been discarded in arriving at equations 30. Note that these expressions differ from those obtained for the flap-lag-pitch sequence by only a single term in each of the shear strains. Proceeding as before, the generalized elastic forces S_u, S_v, S_w, and S_ϕ are given by

$$S_u = - s_1'$$

$$S_v = s_4'' + s_8''$$

$$S_w = s_5'' - s_9'$$

$$S_\phi = s_2'' - s_3' + s_6 - s_7 \tag{31}$$

and the boundary terms B_V by

$$\begin{aligned} B_V = {} & s_1 \delta u\Big|_0^R - (s_4' + s_8')\ \delta v\Big|_0^R + (s_4 + s_8)\ \delta v'\Big|_0^R + s_5 \delta w'\Big|_0^R \\ & + (s_9 - s_5')\ \delta w\Big|_0^R + (s_7 - s_2' + s_3)\ \delta\phi\Big|_0^R + s_2 \delta\phi'\Big|_0^R \end{aligned} \tag{32}$$

where s_1 to s_6 have the same definition as given in equations 20 and s_7 to s_9 are defined by

$$s_7 = GJ\phi' + v''w'GD_2$$

$$s_8 = w'\phi'GD_2$$

$$s_9 = v''\phi'GD_2 \tag{33}$$

KINETIC ENERGY

The expression for the kinetic energy of the blade in terms of the velocity of an arbitrary mass point of the blade is given by (reference 15)

$$T = \tfrac{1}{2}\int_0^R \iint_A \rho \frac{d\bar{r}_1}{dt} \cdot \frac{d\bar{r}_1}{dt}\, d\eta\, d\zeta\, dx \tag{34}$$

and its first variation, integrated between t_o and t_1, is given by

$$\int_{t_o}^{t_1} \delta T = \int_{t_o}^{t_1}\int_0^R \iint_A \rho \frac{d\bar{r}_1}{dt} \cdot \delta\frac{d\bar{r}_1}{dt}\, d\eta\, d\zeta\, dx \tag{35}$$

In equation 35, the absolute velocity of the mass point is $\frac{d\bar{r}_1}{dt}$ and is defined by

$$\frac{d\bar{r}_1}{dt} = \dot{\bar{r}}_1 + \bar{\omega} \times \bar{r}_1 \tag{36}$$

where $\bar{\omega}$ is the angular velocity of the XYZ coordinate system (figure 1), and $\bar{r}_1$ is the position vector to the mass point. The angular velocity $\bar{\omega}$ is obtained by projecting Ω along the X, Y, and Z directions and is given by

$$\bar{\omega} = \Omega \sin \beta_{pc} \; \bar{e}_X + \Omega \cos \beta_{pc} \; \bar{e}_Z \tag{37}$$

Assuming the precone angle β_{pc} to be small, the expression for $\bar{\omega}$ can be approximated by

$$\bar{\omega} = \Omega \; \beta_{pc} \; \bar{e}_X + \Omega \; \bar{e}_Z \tag{38}$$

Differentiating $\bar{r}_1$ with respect to time according to equation 36, the absolute velocity of the mass point can be written as

$$\frac{d\bar{r}_1}{dt} = (\dot{x}_1 - y_1\Omega)\bar{e}_X + (\dot{y}_1 + \Omega x_1 - \Omega\beta_{pc}z_1)\bar{e}_Y + (\dot{z}_1 + y_1\Omega\beta_{pc})\bar{e}_Z \tag{39}$$

The generalized inertia forces will now de derived for each of the two transformation sequences which are addressed herein.

Flap-Lag-Pitch Transformation Sequence

Substituting x_1, y_1, and z_1 from equations 13 into 39 and the result into equation 35, integrating by parts over time where necessary, and then integrating over the cross section, the variation of T can be put into the form

$$\delta T = \int_0^R (k_1\delta u - k_1\delta U_F + k_2\delta v' + k_3\delta v - k_4\delta w' - k_5\delta w + k_6\delta\phi)dx \tag{40}$$

where, consistent with the ordering scheme given in Appendix A,

$$k_1 = -m(\ddot{u} - \ddot{U}_F) + 2m\Omega[\dot{v} - e(\dot{\phi} + \dot{\theta}) \sin\theta - e\phi\dot{\theta}\cos\theta]$$

$$+ m\Omega^2(x + u - U_F - ev' \cos\theta - ew' \sin\theta)$$

$$- m\Omega^2\beta_{pc}(w + e \sin\theta + e\phi \cos\theta) + me\ddot{v}' \cos\theta$$

$$+ me[(\ddot{w}' - v'\ddot{\theta} - 2\dot{v}'\dot{\theta}) \sin\theta + (2\dot{w}'\dot{\theta} + w'\ddot{\theta}) \cos\theta]$$

$$k_2 = m\Omega^2 e\phi x \sin\theta - 2me\Omega\dot{v} \cos\theta - me\Omega^2 x \cos\theta$$

$$k_3 = m\Omega^2(v + e\cos\theta - e\phi\sin\theta) - m\ddot{v} + me(\ddot{\phi} + \ddot{\theta}) \sin\theta + 2m\Omega\beta_{pc}\dot{w}$$

$$- 2m\Omega(\dot{u} - \dot{U}_F - e\dot{v}' \cos\theta - e\dot{w}' \sin\theta)$$

$$k_4 = m\Omega^2 e\phi x \cos\theta + 2m\Omega e\dot{v} \sin\theta + m\Omega^2 ex \sin\theta$$

$$k_5 = m\ddot{w} + me(\ddot{\phi} + \ddot{\theta}) \cos\theta + 2m\Omega\beta_{pc}\dot{v} + m\Omega^2\beta_{pc}x$$

$$k_6 = -w'(2m\Omega e\dot{v} \cos\theta + m\Omega^2 ex \cos\theta) - m\Omega^2 e\phi v \cos\theta$$

$$- m\Omega^2\phi(k_{m_2}^2 - k_{m_1}^2) \cos 2\theta - m\Omega^2(k_{m_2}^2 - k_{m_1}^2) \sin\theta \cos\theta$$

$$+ me\phi\ddot{v} \cos\theta + 2m\Omega ev'\dot{v} \sin\theta + m\Omega^2 exv' \sin\theta$$

$$- m\Omega^2 ev \sin\theta + me\phi\ddot{w} \sin\theta + m\Omega^2\beta_{pc} e\phi x \sin\theta$$

$$+ 2m\Omega\Big[e \sin\theta\,(\dot{u} - \dot{U}_F) - (k_{m_2}^2 - k_{m_1}^2)\dot{v}' \sin\theta \cos\theta$$

$$- \dot{w}'(k_{m_2}^2 \sin^2\theta + k_{m_1}^2 \cos^2\theta)\Big] + me\ddot{v} \sin\theta$$

$$- 2m\Omega\beta_{pc} e\dot{w} \sin\theta - mk_m^2(\ddot{\phi} + \ddot{\theta}) - m\Omega^2\beta_{pc} ex \cos\theta$$

$$- me\ddot{w} \cos\theta - 2m\Omega\beta_{pc} e\dot{v} \cos\theta \qquad (41)$$

The sectional properties appearing in equations 41 are defined as follows:

$$m = \iint_A \rho \, d\eta \, d\zeta \qquad me = \iint_A \rho\eta \, d\eta \, d\zeta$$

$$mk_{m_1}^2 = \iint \rho\zeta^2 d\eta \, d\zeta \qquad mk_{m_2}^2 = \iint \rho\eta^2 \, d\eta \, d\zeta$$

$$k_m^2 = k_{m_1}^2 + k_{m_2}^2 \qquad (42)$$

From symmetry of the cross section about the η axis and the anti-symmetry of the warping function, the following integrals have been set to zero:

$$\iint \rho\zeta \, d\eta \, d\zeta = 0 \qquad \iint \rho\eta\zeta \, d\eta \, d\zeta = 0$$

$$\iint \rho\lambda \, d\eta \, d\zeta = 0 \qquad \iint \rho\lambda\eta \, d\eta \, d\zeta = 0 \tag{43}$$

Since U_F is a function of v' and w', the term involving δU_F in equation 40 requires separate treatment. Using equation 10, the second term in equation 40 can be written in the expanded form

$$\int_0^R k_1 \delta U_F \, dx = \int_0^R k_1 \left[\int_0^x (w'\delta w' + v'\delta v')dx \right] dx \tag{44}$$

which can be further rewritten as

$$\int_0^R k_1 \delta U_F \, dx = \int_0^R \left[\int_x^R k_1 \, dx \right] (w'\delta w' + v'\delta v')dx \tag{45}$$

Defining the tension T as

$$T = \int_x^R k_1 \, dx \tag{46}$$

equation 45 can be written as

$$\int_0^R k_1 \delta U_F \, dx = \int_0^R T(w'\delta w' + v'\delta v')dx \tag{47}$$

Integrating equation 40 by parts, the resulting expression can be put into the form

$$\delta T = \int_0^R (I_u \delta u + I_v \delta v + I_w \delta w + I_\phi \delta\phi)dx + B_T \tag{48}$$

where the generalized inertia forces I_u, I_v, I_w, and I_ϕ are given by

$$I_u = k_1 = -T'$$

$$I_v = -k_2' + k_3 + (Tv')'$$

$$I_w = k_4' - k_5 + (Tw')'$$

$$I_\phi = k_6 \tag{49}$$

and the boundary terms B_T by

$$B_T = (k_2 - Tv')\delta v \Big|_0^R - (k_4 + Tw')\delta w \Big|_0^R \tag{50}$$

Lag-Flap-Pitch Transformation Sequence

Proceeding as in the previous section, this time using x_1, y_1, and z_1 from equations 29, the generalized inertia forces I_u, I_v, I_w, and I_ϕ and the boundary terms B_T for a lag-flap-pitch sequence are identical to those obtained in the previous section for a flap-lag-pitch sequence. Formally, the generalized inertia forces corresponding to the two transformation sequences addressed are different. However, because of the ordering scheme employed (see Appendix A) the differences, which occur in the higher order terms, disappear.

VIRTUAL WORK DUE TO MATERIAL DAMPING

The virtual work due to the dissipative forces associated with structural (material) damping can be expressed in the form

$$\delta W_D = \sum_{k=1}^{4} Q_{D_k} \, \delta q_k \tag{51}$$

where Q_{D_k} is the generalized damping force associated with the k^{th} dependent variable and δq_k is the variation of the k^{th} dependent variable. In the present development the generalized damping forces accounting for the dissipation of energy due to material damping will be taken to be those consistent with the assumption of a material which exhibits a linear visco-elastic behavior. This theory (see, for example, references 19 and 20) assumes that the stresses are linear functions of the strains and strain rates. Such a behavior is analogous to a spring and a dashpot in parallel, and a model which exhibits such a behavior is often termed a Kelvin-Voigt solid in the literature (references 20 and 21). A model of this type was used in reference 22 for a rotating beam. For the stresses and strains of interest herein, these constitutive relations have the form

$$\begin{aligned} \tau_{xx} &= E\gamma_{xx} + E^{*}\dot{\gamma}_{xx} \\ \tau_{x\eta} &= G\gamma_{x\eta} + G^{*}\dot{\gamma}_{x\eta} \\ \tau_{x\zeta} &= G\gamma_{x\zeta} + G^{*}\dot{\gamma}_{x\zeta} \end{aligned} \tag{52}$$

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where E and G are Young's modulus and the shear modulus, respectively, and E^* and G^* are coefficients which take into account internal damping of the material in tension and shear, respectively. The first term on the right hand side of each of equations 52 contributes to the usual elastic strain energy and have already been treated in an earlier section. Considering only the dissipative terms in equations 52, the virtual work of the structural dissipative forces can be written as

$$\delta W_D = -\int_0^R E^* \iint_A \dot{\gamma}_{xx}\, \delta\gamma_{xx}\, d\eta\, d\zeta\, dx - \int_0^R G^* \iint_A (\dot{\gamma}_{x\eta}\, \delta\gamma_{x\eta} + \dot{\gamma}_{x\zeta}\, \delta\gamma_{x\zeta}) d\eta\, d\zeta\, dx \tag{53}$$

The result given in equation 53 is general. However, because of the lack of knowledge as to the distribution of damping, only the direct damping terms are generally retained in practice. Thus, off-diagonal terms accounting for damping coupling between the dependent variables which arise from equation 53 are taken to be zero and only the direct damping terms associated with the dependent variables are retained. In addition to adopting this expedient in the present development, it will also be assumed that a first approximation to the direct damping terms can be obtained by retaining only the linear damping terms in the final equations of motion. Thus, it is sufficient to retain terms up to only first degree in the expressions for the strains. To first degree the resulting strain expressions will be the same for both of the transformation sequences, specifically

$$\gamma_{xx} = u' - \lambda\phi'' + (\eta^2 + \zeta^2)\phi'\theta'_{pt} - v''(\eta\cos\theta - \zeta\sin\theta)$$

$$- w''(\eta\sin\theta + \zeta\cos\theta)$$

$$\gamma_{x\eta} = -\hat{\zeta}\phi'$$

$$\gamma_{x\zeta} = \hat{\eta}\phi' \tag{54}$$

Substituting equations 54 into equation 53, integrating over the cross section, and retaining only the linear direct damping terms leads to

$$\delta W_D = -\int_0^R (d_1\delta u' + d_2\delta\phi'' + d_3\delta\phi' + d_4\delta v'' + d_5\delta w'' + d_6\delta\phi')dx \tag{55}$$

where

$$d_1 = E^*A\dot{u}'$$

$$d_2 = E^*C_1\dot{\phi}''$$

$$d_3 = E^*B_1\theta'^2_{pt}\,\dot{\phi}'$$

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$$d_4 = E^*(I_{\zeta\zeta}\cos^2\theta + I_{\eta\eta}\sin^2\theta)\dot{v}''$$

$$d_5 = E^*(I_{\zeta\zeta}\sin^2\theta + I_{\eta\eta}\cos^2\theta)\dot{w}''$$

$$d_6 = G^* J\dot{\phi}' \tag{56}$$

Integrating equation 55 by parts, the generalized damping forces Q_{D_u}, Q_{D_v}, Q_{D_w}, and Q_{D_ϕ} become

$$Q_{D_u} = d_1' = (E^* A\dot{u}')'$$

$$Q_{D_v} = -d_4'' = -\left[E^*(I_{\zeta\zeta}\cos^2\theta + I_{\eta\eta}\sin^2\theta)\dot{v}''\right]''$$

$$Q_{D_w} = -d_5'' = -\left[E^*(I_{\zeta\zeta}\sin^2\theta + I_{\eta\eta}\cos\theta)\dot{w}''\right]''$$

$$Q_{D_\phi} = -d_2'' + d_3' + d_6' = -(E^* C_1\dot{\phi}'')'' + (E^* B_1\theta_{pt}'^2\dot{\phi}')' + (G^* J\dot{\phi}')' \tag{57}$$

and the boundary terms $B_{\delta W_D}$ become

$$B_{\delta W_D} = -d_1\delta u\Big|_0^R + d_4'\delta v\Big|_0^R - d_4\delta v'\Big|_0^R + d_5'\delta w\Big|_0^R$$
$$-d_5\delta w'\Big|_0^R - (d_3 - d_2' + d_6)\delta\phi\Big|_0^R - d_2\delta\phi'\Big|_0^R \tag{58}$$

GENERALIZED AERODYNAMIC FORCES

The aerodynamic forces will be generated from two-dimensional, incompressible, quasi-steady, strip theory in which only the velocity components perpendicular to the spanwise axis of the deformed blade (the x_3 axis) are assumed to influence the aerodynamic loading. Account is taken of the pulsating free-stream velocity $V(t)$ associated with a rotating blade by employing Greenberg's extension of Theodorsen's unsteady theory (reference 23) for determining the aerodynamic lift and pitching moment acting on the blade. The resulting expressions are specialized to the case of quasi-steady flow by setting Theodorsen's circulation function to unity. Classical blade element momentum theory is used to calculate the steady flow induced by the rotor.

In the present application of Greenberg's theory, the airfoil is taken to be pivoted in pitch about the aerodynamic center at the quarter chord and to be executing harmonic motions in pitch ($\varepsilon(t)$) and plunge ($\dot{h}(t)$) while immersed in a pulsating airstream $V(t)$, as shown in figure 5. The lift and moment acting on an elemental section of the blade may be expressed in terms of the circulatory and noncirculatory components as

$$L = L_C + L_{NC}$$

$$M = M_C + M_{NC} \tag{59}$$

Assuming that the blade elastic axis is coincident with the aerodynamic center at the quarter chord, the individual components of equation 59 follow from reference 23 and can be written as

$$L_{NC} = \tfrac{1}{2}\rho a \frac{c^2}{4} (\ddot{h} + V\dot{\varepsilon} + \dot{V}\varepsilon + \frac{c}{4} \ddot{\varepsilon}) \tag{60a}$$

$$L_C = \tfrac{1}{2}\rho acV(\dot{h} + V\varepsilon + \frac{c}{2} \dot{\varepsilon}) \tag{60b}$$

$$M_{NC} = -\tfrac{1}{2}\rho ac(\frac{c}{4})^2(\dot{V}\varepsilon + \ddot{h} + \frac{3c}{8} \ddot{\varepsilon}) \tag{60c}$$

$$M_C = -\tfrac{1}{2}\rho ac(\frac{c}{4})^2\ 2V\dot{\varepsilon} \tag{60d}$$

In the course of arriving at the circulatory terms in equations 60, the quasi-steady approximation has been introduced by setting the reduced frequency k to zero, in consequence of which Theodorsen's circulation function $C(k)$ assumes the value of unity. The noncirculatory lift and moment are associated with apparent mass forces and are oftentimes discarded in rotor blade applications. Note that Greenberg's modification (i.e., a pulsating stream in which $\dot{V} \neq 0$) appears only in the noncirculatory expressions for the lift and moment. Hence, if one assumes, a priori, that apparent mass forces will be neglected there is no Greenberg modification.

The lifts and moments given in equations 60 must now be expressed in terms of U_R, U_T, and U_P, the radial, tangential, and perpendicular velocity components relative to a point on the elastic axis of the airfoil (figure 6). Now the expression in the parentheses of equation 60a for L_{NC} is the downward acceleration of the mid-chord point of the airfoil, and the expression in the parentheses of equation 60b for L_C is the downward velocity of the three-quarter-chord point of the airfoil. Since U_P is the relative velocity component perpendicular to the quarter-chord, the sectional lifts can also be written as

$$L_{NC} = \tfrac{1}{2}\rho a \frac{c^2}{4}\left(-\dot{U}_P + \frac{c}{4}\ddot{\varepsilon}\right) \tag{61a}$$

$$L_C = \tfrac{1}{2}\rho acU\left(-U_P + \frac{c}{2}\dot{\varepsilon}\right) \tag{61b}$$

where $V(t)$, appearing outside the parentheses of equation 60b, has been approximated by the resultant of only the tangential and perpendicular velocity components and is given by

$$V \simeq U \simeq \sqrt{U_T^2 + U_P^2} \tag{62}$$

As indicated in figure 7, the noncirculatory lift acts normal to the section chordline* and the circulatory lift acts normal to the resultant velocity U. The profile drag force acts parallel to U and is given by

$$D = \tfrac{1}{2}\rho ac \frac{C_{d_o}}{a} U^2 \tag{63}$$

where C_{d_o} is the (constant) profile drag coefficient.

The components of the aerodynamic forces in the directions of the y_3 and z_3 axes are given by

*A portion of L_{NC} acts at the 3/4-chord point and another at the 1/2-chord point. However, the resultant of these two components is shown along the z_3 axis in figure 7 only for pictorial convenience.

$$F_{y_3} = - L_C \sin \alpha - D \cos \alpha \qquad (64a)$$

$$F_{z_3} = L_C \cos \alpha + L_{NC} - D \sin \alpha \qquad (64b)$$

where, from figure 7,

$$\sin \alpha = U_P/U$$

$$\cos \alpha = U_T/U \qquad (65)$$

and U is given by equation 62. .he aerodynamic force in the x_3 direction is given by F_{x_3} and is a profile drag force which is a function of the radial velocity component U_R. Following usual practice, this force component is assumed to have a negligible effect on stability and F_{x_3} is taken to be zero. Substituting equations 61, 63, and 65 into equations 64 and assuming that U_P/U_T and C_{d_o}/a are negligible compared to unity leads to

$$F_{y_3} = \tfrac{1}{2}\rho ac\left[U_P{}^2 - \frac{c}{2} U_P\dot{\varepsilon} - \frac{C_{d_o}}{a} U_T{}^2\right] \qquad (66a)$$

$$F_{z_3} = \tfrac{1}{2}\rho ac\left[- U_PU_T + \frac{c}{2} U_T\dot{\varepsilon} - \frac{c}{4} \dot{U}_P + (\frac{c}{4})^2 \ddot{\varepsilon}\right] \qquad (66b)$$

The noncirculatory and circulatory moments given in equations 60c and 60d can be written in terms of U_T, U_P, and ε and assume the form

$$M_{NC} = -\tfrac{1}{2}\rho ac \left(\frac{c}{4}\right)^2 \left[-\dot{U}_P - U_T\dot{\varepsilon} + \frac{3c}{8}\ddot{\varepsilon} \right] \tag{67a}$$

$$M_C = -\tfrac{1}{2}\rho ac \left(\frac{c}{4}\right)^2 2\, U_T\dot{\varepsilon} \tag{67b}$$

from which the total pitching moment M_ϕ is given by the sum of equations 67a and 67b as

$$M_\phi (= M_{x_3}) = -\tfrac{1}{2}\rho ac \left(\frac{c}{4}\right)^2 \left[U_T\dot{\varepsilon} - \dot{U}_P + \frac{3c}{8}\ddot{\varepsilon} \right] \tag{68}$$

It should be remarked here that for the special case involving only coupled flapwise and edgewise bending which is often addressed in the literature, the quasi-steady approximation to the aerodynamic loading is usually taken to be completely determined by the square of the resultant of U_T and U_P acting at the quarter chord of the section. It is interesting to note that the quasi-steady approximation of the lift arrived at by setting $C(k) = 1$ in the general unsteady aerodynamic expressions of Theodorsen and discarding all the noncirculatory terms contains an additional term involving $\dot{\varepsilon}$ (see equation 61b) which does not arise when proceeding in the other manner.

The virtual work of the aerodynamic forces can be written as

$$\delta W_A = \int_0^R (A_u\delta u + A_v\delta v + A_w\delta w + A_\phi\delta\phi)dx \tag{69}$$

where the generalized aerodynamic forces A_u, A_v, A_w, and A_ϕ are given by

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$$
\begin{Bmatrix} A_u \\ A_v \\ A_w \end{Bmatrix} = [T]^T \begin{Bmatrix} 0 \\ F_{y_3} \\ F_{z_3} \end{Bmatrix} \tag{70}
$$

and

$$
A_\phi = M_\phi (= M_{x_3}) \tag{71*}
$$

where [T] is the rotational transformation matrix which relates the coordinate axes of the deformed and undeformed blade and F_{x_3} has been set to zero. The explicit form of the matrix depends on the rotational sequence employed in arriving at the matrix [T], as already mentioned. In order to obtain explicit expressions for the generalized aerodynamic forces, the quantities F_{y_3}, F_{z_3}, and M_{x_3} must be known in terms of the dependent variables u, v, w, and ϕ, and the geometric pitch angle θ. This requires that U_T, U_P, and $\dot{\varepsilon}$ first be obtained in terms of these quantities.

The resultant velocities seen by a point on the elastic axis of the blade in the deformed and the undeformed coordinate systems are related according to

$$
\bar{V}_{x_3 y_3 z_3} = [T]\, \bar{V}_{XYZ} \tag{72}
$$

where, from figure 6,

---

*No transformation is needed for A_ϕ since the rotation ϕ is about the x_3 axis.

$$\bar{V}_{x_3 y_3 z_3} = U_R \bar{e}_{x_3} - U_T \bar{e}_{y_3} - U_P \bar{e}_{z_3} \tag{73}$$

and $\bar{V}_{XYZ}$, the total relative velocity (aerodynamic + dynamic) of a point on the elastic axis of the blade, is given by

$$\bar{V}_{XYZ} = \left[\bar{V}_a - \frac{d\bar{r}_1}{dt}\right]_{XYZ} \tag{74}$$

The aerodynamic velocity components seen by a blade element are shown in figure 8. Using figures 1 and 8, the flow relative to the blade due to forward flight velocity V and induced flow v_i can be written as

$$(\bar{V}_a)_{XYZ} = (\mu\Omega R \cos\psi + \Omega R\lambda\beta_{pc})\bar{e}_X - \mu\Omega R \sin\psi \; \bar{e}_Y$$

$$+ (\Omega R\lambda - \mu\Omega R\beta_{pc} \cos\psi)\bar{e}_Z \tag{75}$$

where the advance ratio μ and inflow ratio λ are defined by

$$\mu\Omega R = V \cos\alpha_s$$

$$\lambda\Omega R = V \sin\alpha_s - v_i \tag{76}$$

The induced velocity v_i is calculated by equating the integrated thrust to the thrust from momentum theory using the relations

$$v_i = \frac{C_T \Omega R}{2\sqrt{\mu^2 + \lambda^2}} \tag{77}$$

$$C_T = \frac{b}{2\pi} \frac{1}{\pi\rho\Omega^2 R^4} \int_0^{2\pi} \int_0^R A_{w_o} \, d\psi \tag{78}$$

where A_{w_o} is the steady-state value of A_w obtained from equation 70.
On the elastic axis

$$\bar{r}_1 = (x + u - U_F)\, \bar{e}_X + v\bar{e}_Y + w\bar{e}_Z \tag{79}$$

and is independent of the transformation sequence. Using equation 79 in conjunction with equations 36 and 38 the dynamic velocity of a point on the elastic axis taken with respect to the XYZ axis system is given by

$$\left[\frac{d\bar{r}_1}{dt}\right]_{XYZ} = (\dot{u} - \dot{U}_F - \Omega v)\bar{e}_X + \left[\dot{v} - \Omega\beta_{pc} w + \Omega(x + u - U_F)\right]\bar{e}_Y$$

$$+ (\dot{w} + \Omega\beta_{pc} v)\bar{e}_Z \tag{80}$$

Combining equations 75 and 80 according to equation 74 gives the total velocity seen by a point on the elastic axis as

$$\bar{V}_{XYZ} = \left[\mu\Omega R \cos\psi + \Omega R\lambda\beta_{pc} - \dot{u} + \dot{U}_F + \Omega v\right]\bar{e}_X$$

$$+ \left[\Omega\beta_{pc}\, w - \dot{v} - (x + u - U_F) - \mu\Omega R \sin\psi\right]\bar{e}_Y$$

$$+ \left[\Omega R\lambda - \mu\Omega R\beta_{pc} \cos\psi - \dot{w} - \Omega\beta_{pc} v\right]\bar{e}_Z \tag{81}$$

The quantity $\dot{\varepsilon}$ appearing in equations 66 and 68 is the angular velocity of the blade section about the local x_3 axis and, consistent with the present notation, can be written as $\dot{\varepsilon}_{x_3}$. It can be regarded as composed of three parts: the first part arising from the rigid-body angular velocity of the hub in space, the second part arising from the control inputs and kinematic couplings, and the third part arising from the angular velocity associated with the elastic deformations. Since the only rigid-body angular velocity of the hub in the present mathematical model is that due to the blade rotational speed Ω, the first contribution to $\dot{\varepsilon}_{x_3}$ is obtained from

$$\left\{\begin{matrix} \dot{\varepsilon}_{x_3} \\ \\ \dot{\varepsilon}_{y_3} \\ \\ \dot{\varepsilon}_{z_3} \end{matrix}\right\}_{\Omega} = [T] \left\{\begin{matrix} \Omega\beta_{pc} \\ \\ 0 \\ \\ \Omega \end{matrix}\right\} \tag{82}$$

The contribution of the control inputs and kinematic coupling to the rigid-body pitching motion of the section is given by

$$(\dot{\varepsilon}_{x_3})_{c\&kc} = \dot{\theta} \tag{83}$$

The contribution to $\dot{\varepsilon}_{x_3}$ associated with the elastic deformation is obtained by projecting the component angular velocities $\dot{\phi}$, $\dot{v}'$, and $\dot{w}'$ onto the x_3 axis. Formally, all three components can be obtained by replacing the derivatives of the Euler angles β', ζ', and θ' in the initial step of the derivation of the nonlinear curvature expressions in reference 14 by $\dot{\beta}$, $\dot{\zeta}$, and $\dot{\theta}$ and then continuing the development. This is equivalent to replacing v'', w'', and ϕ' by $\dot{v}'$, $\dot{w}'$, and $\dot{\phi}$, respectively, in the curvature expressions given in reference 14. In the next two sections, explicit expressions for U_T, U_P, and $\dot{\varepsilon}$ will be developed.

Flap-Lag-Pitch Transformation Sequence

For this transformation sequence, the tangential and perpendicular velocity components U_T and U_P are obtained from equations 72 and 73 using $[T_{FLP}]$ from equation 11 and $\bar{V}_{XYZ}$ from equation 81 and, to second-degree in the dependent variables, have the form

$$
\begin{aligned}
U_T = & \left[v'(\cos\theta - \phi\sin\theta) + w'(\sin\theta + \phi\cos\theta)\right]\left[\Omega R\lambda\beta_{pc} + \mu\Omega R\cos\psi\right] \\
& + (v'\cos\theta + w'\sin\theta)(\Omega v - \dot{u}) \\
& - \left[\phi\sin\theta - \left(1 - \frac{v'^2}{2} - \frac{\phi^2}{2}\right)\cos\theta\right]\left[\Omega x + \mu\Omega R\sin\psi\right] \\
& + \left[\Omega u + \dot{v} - \Omega\beta_{pc}w\right]\left[\cos\theta - \phi\sin\theta\right] - \Omega U_F\cos\theta \\
& - \left[\mu\Omega R\beta_{pc}\cos\psi - \Omega R\lambda\right]\left[\left(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}\right)\sin\theta + (\phi - v'w')\cos\theta\right] \\
& + \left[\dot{w} + \Omega\beta_{pc}v\right]\left[\sin\theta + \phi\cos\theta\right]
\end{aligned}
\qquad (84a)
$$

$$
\begin{aligned}
U_P = & \left[\mu\Omega R \cos\psi + \Omega R\lambda\beta_{pc}\right]\left[w'(\cos\theta - \phi\sin\theta) - v'(\sin\theta + \phi\cos\theta)\right] \\
& + \left[w'\cos\theta - v'\sin\theta\right](\Omega v - \dot{u}) + \Omega U_F \sin\theta \\
& - \left[\Omega x + \mu\Omega R\sin\psi\right]\left[(1 - \frac{v'^2}{2} - \frac{\phi^2}{2})\sin\theta + \phi\cos\theta\right] \\
& + \left[\Omega\beta_{pc} w - \dot{v} - \Omega u\right]\left[\sin\theta + \phi\cos\theta\right] \\
& + \left[\mu\Omega R\beta_{pc}\cos\psi - \Omega R\lambda\right]\left[(v'w' - \phi)\sin\theta + (1 - \frac{w'^2}{2} - \frac{\phi^2}{2})\cos\theta\right] \\
& + \left[\dot{w} + \Omega\beta_{pc} v\right]\left[\cos\theta - \phi\sin\theta\right] \qquad (84b)
\end{aligned}
$$

Using equation 82 with $[T] = [T_{FLP}]$ from equation 11, the sectional pitching velocity due to Ω is found to be

$$(\dot{\epsilon}_{x_3})_\Omega = \Omega\beta_{pc}(1 - \frac{v'^2}{2} - \frac{w'^2}{2}) + \Omega w' \qquad (85)$$

The sectional pitching velocity associated with the control inputs and kinematic coupling is given by

$$(\dot{\epsilon}_{x_3})_{c\&kc} = \dot{\theta}_{kc} + \theta_{1c}\Omega\sin\psi - \theta_{1s}\Omega\cos\psi \qquad (86)$$

Replacing ϕ' by $\dot{\phi}$ and w'' by $\dot{w}'$ in equation 12 (while discarding the pretwist), the sectional pitching velocity due to the elastic deformation is given by

$$(\dot{\varepsilon}_{x_3})_{\text{deformation}} = \dot{\phi} - \dot{w}'v' \tag{87}$$

Combining equations 85, 86, and 87, the total sectional pitching velocity is

$$\dot{\varepsilon}(= \dot{\varepsilon}_{x_3}) = \Omega\beta_{pc}(1 - \frac{v'^2}{2} - \frac{w'^2}{2}) + \Omega w' + \dot{\phi} - \dot{w}'v'$$

$$+ \dot{\theta}_{kc} + \theta_{1c}\Omega \sin \psi - \theta_{1s}\Omega \cos \psi \tag{88}$$

Equations 84 and 88 in combination with equations 66 and 68 are sufficient to obtain the generalized aerodynamic forces from equations 70 and 71.

Lag-Flap-Pitch Transformation Sequence

For this transformation sequence, the tangential and perpendicular velocity components U_T and U_P are obtained from equations 72 and 73 using $[T_{LFP}]$ from equation 27 and $\bar{V}_{XYZ}$ from equation 81, and, to second degree in the dependent variables, have the form

$$U_T = \left[\mu\Omega R \cos \psi + \Omega R\lambda\beta_{pc}\right]\left[v'(\cos \theta - \phi \sin \theta) + w'(\sin \theta + \phi \cos \theta)\right]$$

$$+ \left[\Omega v - \dot{u}\right]\left[v' \cos \theta + w' \sin \theta\right] + \left[\Omega x + \mu\Omega R \sin \psi\right] \cdot$$

$$\left[- (\phi + v'w') \sin \theta + (1 - \frac{v'^2}{2} - \frac{\phi^2}{2}) \cos \theta\right] + \left[\dot{v} + \Omega u - \Omega\beta_{pc} w\right] \cdot$$

$$\left[\cos \theta_{pt} - \phi \sin \theta\right] - \Omega U_F \cos \theta + \left[\mu\Omega R\beta_{pc} \cos \psi - \Omega R\lambda\right] \cdot$$

$$\left[(1 - \frac{w'^2}{2} - \frac{\phi^2}{2}) \sin\theta + \phi\cos\theta\right] + \left[\dot{w} + \Omega\beta_{pc} v\right]\left[\sin\theta + \phi\cos\theta\right] \tag{89a}$$

$$\begin{aligned}
U_P = &\left[\mu\Omega R \cos\psi + \Omega R\lambda\beta_{pc}\right]\left[v'(\cos\theta - \phi\sin\theta) - v'(\sin\theta + \phi\cos\theta)\right] \\
&+ \left[\Omega v - \dot{u}\right]\left[w'\cos\theta - v'\sin\theta\right] - \left[\mu\Omega R\sin\psi + \Omega x\right] \bullet \\
&\left[(\phi + v'w')\cos\theta + (1 - \frac{v'^2}{2} - \frac{\phi^2}{2})\sin\theta\right] + \left[\Omega\beta_{pc} w - \dot{v} - \Omega u\right] \bullet \\
&\left[\phi\cos\theta + \sin\theta\right] + \Omega U_F \sin\theta + \left[\mu\Omega R\beta_{pc}\cos\psi - \Omega R\lambda\right] \bullet \\
&\left[(1 - \frac{w'^2}{2} - \frac{\phi^2}{2})\cos\theta - \phi\sin\theta\right] + \left[\dot{w} + \Omega\beta_{pc} v\right]\left[\cos\theta - \phi\sin\theta\right]
\end{aligned} \tag{89b}$$

Using equation 82 with $[T] = [T_{LFP}]$ from equation 27 and replacing ϕ' by $\dot{\phi}$ and v'' by $\dot{v}'$ in equation 28 (while discarding the pretwist), the resultant pitching velocity of the section is

$$\begin{aligned}
\dot{\varepsilon}\ (= \dot{\varepsilon}_{x_3}) = &\ \Omega\beta_{pc}(1 - \frac{v'^2}{2} - \frac{w'^2}{2}) + \Omega w' + \dot{\phi} + \dot{v}'w' \\
&+ \dot{\theta}_{kc} + \theta_{1c}\Omega\sin\psi - \theta_{1s}\Omega\cos\psi
\end{aligned} \tag{90}$$

Equations 89 and 90 in combination with equations 66 and 68 are sufficient to obtain the generalized aerodynamic forces from equations 70 and 71.

SUMMARY OF EQUATIONS

Expressions for δT, δV, and δW have been obtained above for two rotational transformation sequences. Substituting these expressions and their associated boundary terms into equation 2 there results an expression of the form

$$\int_{t_o}^{t_1}\left\{\int_0^R [(\quad)\delta u + (\quad)\delta v + (\quad)\delta w + (\quad)\delta\phi]dx + B\right\}dt = 0 \qquad (91)$$

For arbitrary, admissible variations δu, δv, δw, and $\delta\phi$, the four expressions in parentheses must vanish individually as must the assembly of boundary terms denoted by B. The first condition will yield the four governing nonlinear differential equations for u, v, w, and ϕ and the second condition will give the associated boundary conditions at the ends of the beam. Since the control inputs are assumed to be known functions of time and the dynamics of the kinematic control mechanism are not considered, the equations associated with the control and kinematic motions will not appear. The governing equations of motion and boundary conditions are summarized below for each of the two sequences considered.

Flap-Lag-Pitch Transformation Sequence

Extension:

$$m(\ddot{u} - \ddot{U}_F) - me(\ddot{v}'\cos\theta + \ddot{w}'\sin\theta) - 2m\Omega[\dot{v} - e(\dot{\phi} + \dot{\theta})\sin\theta - e\dot{\phi}\theta\cos\theta]$$

$$- m\Omega^2(x + u - U_F - ev'\cos\theta - ew'\sin\theta) + m\Omega^2\beta_{pc}(w + e\sin\theta + e\phi\cos\theta)$$

$$- \{EA[u' + k_A^2\phi' \theta'_{pt} - e_A(v'' + \phi w'') \cos\theta + e_A(\phi v'' - w'') \sin\theta]$$

$$+ E^* A\dot{u}'\}' = A_u \tag{92a}$$

Chordwise bending:

$$m\ddot{v} - me(\ddot{\phi} + \ddot{\theta}) \sin\theta - 2m\Omega\beta_{pc}\dot{w} - m\Omega^2(v + e\cos\theta - e\phi\sin\theta)$$

$$+ \{me[\Omega^2 x(\phi\sin\theta - \cos\theta) - 2\Omega\dot{v}\cos\theta] + \phi' w'' GD_2\}'$$

$$+ 2m\Omega(\dot{u} - U_F - e\dot{v}'\cos\theta - e\dot{w}'\sin\theta) - (Tv')'$$

$$+ \{EAe_A u'(\phi\sin\theta - \cos\theta) - EB_2\phi' \theta'_{pt}\cos\theta$$

$$- EC_2\phi''\sin\theta + w''[(EI_{\zeta\zeta} - EI_{\eta\eta})(\sin\theta\cos\theta + \phi\cos 2\theta)]$$

$$+ v''[EI_{\eta\eta}(\sin^2\theta + \phi\sin 2\theta) + EI_{\zeta\zeta}(\cos^2\theta - \phi\sin 2\theta)]$$

$$+ E^*(I_{\zeta\zeta}\cos^2\theta + I_{\eta\eta}\sin^2\theta)\dot{v}''\}'' = A_v \tag{92b}$$

Flapwise bending:

$$m\ddot{w} + me(\ddot{\phi} + \ddot{\theta})\cos\theta + 2m\Omega\beta_{pc}\dot{v} - (Tw')'$$

$$- \{me[\Omega^2 x(\phi\cos\theta + \sin\theta) + 2\Omega\dot{v}\sin\theta]\}'$$

$$+ \{EC_2\phi''\cos\theta - EB_2\phi' \theta'_{pt}\sin\theta - EAe_A u'(\phi\cos\theta + \sin\theta)$$

$$+ w''[EI_{\eta\eta} \cos^2 \theta + EI_{\zeta\zeta} \sin^2 \theta + \phi(EI_{\zeta\zeta} - EI_{\eta\eta}) \sin 2\theta]$$

$$+ v''[(EI_{\zeta\zeta} - EI_{\eta\eta})(\sin\theta \cos\theta + \phi \cos 2\theta)] - \phi' v' GD_2$$

$$+ E^*(I_{\zeta\zeta} \sin^2 \theta + I_{\eta\eta} \cos^2 \theta)\dot{w}''\}'' = A_w - m\Omega^2 \beta_{pc} x \qquad (92c)$$

Torsion:

$$mk_m^2(\ddot{\phi} + \ddot{\theta}) + m\Omega^2\phi(k_{m_2}^2 - k_{m_1}^2) \cos 2\theta$$

$$+ me[\Omega^2 x(w' \cos\theta - v' \sin\theta) - (\ddot{v} - \Omega^2 v) \sin\theta + \ddot{w} \cos\theta]$$

$$+ m\Omega^2 e\phi(v \cos\theta - x\beta_{pc} \sin\theta)$$

$$- 2m\Omega[e \sin\theta\ (\dot{u} - \dot{U}_F) - (k_{m_2}^2 - k_{m_1}^2)\dot{v}' \sin\theta \cos\theta$$

$$- \dot{w}'\ (k_{m_2}^2 \sin^2\theta + k_{m_1}^2 \cos^2\theta) - e\beta_{pc}(\dot{v} \cos\theta + \dot{w} \sin\theta)]$$

$$+ 2m\Omega e\dot{v}(w' \cos\theta - v' \sin\theta) - me\phi(\ddot{v} \cos\theta + \ddot{w} \sin\theta)$$

$$+ \{EC_1\phi'' + EC_2[w'' \cos\theta - v'' \sin\theta - \phi(w'' \sin\theta + v'' \cos\theta) + E^*C_1\dot{\phi}''\}''$$

$$- \{EAk_A^2 u'(\theta'_{pt} + \phi') + EB_1\theta'^{\,2}_{pt}\ \phi' + GJ\phi' - v'w''GD_2$$

$$+ EB_2[\theta'_{pt}(\phi v'' - w'') \sin\theta - \phi' v'' \cos\theta - \phi' w'' \sin\theta$$

$$- \theta'_{pt}(v'' + \phi w'') \cos\theta] + E^*B_1\theta'_{pt}{}^2\dot{\phi}' + G^*J\dot{\phi}'\}'$$

$$+ (EI_{\zeta\zeta} - EI_{\eta\eta})[v''w'' \cos 2\theta + (w''^2 - v''^2) \sin\theta \cos\theta]$$

$$+ EB_2\phi'\ \theta'_{pt}(v'' \sin\theta - w'' \cos\theta) - EC_2\phi''(v'' \cos\theta + w'' \sin\theta)$$

$$+ EAe_A u'(v'' \sin\theta - w'' \cos\theta) = M_\phi - m\Omega^2\beta_{pc} ex \cos\theta$$

$$- m\Omega^2(k_{m_2}^2 - k_{m_1}^2) \sin\theta \cos\theta \tag{92d}$$

The assembled collection of boundary terms denoted by B is given by

$$B = B_T - B_V + B_{\delta W_D} \tag{93}$$

and the requirement of the vanishing of the individual variational components leads to the relations

$$(s_1 + d_1)\delta u \Big|_0^R = 0$$

$$(k_2 - Tv' - s_9 + s_4' + d_4')\delta v \Big|_0^R = 0$$

$$(s_4 + d_4)\delta v' \Big|_0^R = 0$$

$$(k_4 + Tw' - s_5' - s_8' - d_5')\delta w \Big|_0^R = 0$$

$$(s_5 + s_8 + d_5)\delta w' \Big|_0^R = 0$$

$$(s_7 + s_3 - s_2' + d_3 - d_2' + d_6)\delta\phi \Big|_0^R = 0$$

$$(s_2 + d_2)\delta\phi' \Big|_0^R = 0 \tag{94}$$

from which the boundary conditions can be identified.

The tension T appearing in equations 92 and 94 is given to second order by

$$T = \int_x^R m\Big[-(\ddot{u} - \ddot{U}_F) + e(\ddot{v}' \cos\theta + \ddot{w}' \sin\theta) + 2\Omega[\dot{v} - e(\dot{\phi} + \dot{\theta})\sin\theta]$$

$$+ \Omega^2(x + u - U_F - ev' \cos\theta - ew' \sin\theta)$$

$$- \Omega^2\beta_{pc}(w + e \sin\theta)\Big]\, dx \tag{95}$$

The terms U_F and $\ddot{U}_F$ in the expression for T given in equation 95 lead to third-degree nonlinear terms when T is substituted into equations 92 and 94 and can be discarded. Also, after substituting for T in these equations only resulting terms which are consistent with the ordering scheme adopted in Appendix A should be retained. Using the result given in equation 95 in combination with the extensional equation of motion given in equation 92a (with damping and   set to zero), an alternative definition of T can be given as

$$T = EA[u' + k_A^2\phi'\,\theta_{pt}' - e_A(v'' + \phi w'')\cos\theta + e_A(\phi v'' - w'')\sin\theta] \tag{96}$$

As indicated earlier, the generalized aerodynamic forces A_u, A_v, A_w, and A_ϕ

are obtained from equations 70 and 71 using equations 84 and 88 in combination with equations 66 and 68 and retaining terms through second degree in the dependent variables. Because of the generality of the present development these second-degree expressions are extremely lengthy and will not be shown. For simplicity, comparison of the present aerodynamic results with those existing in the literature will be done using U_T, U_P, and $\dot{\varepsilon}$, which are the primary ingredients of the generalized aerodynamic forces.

Lag-Flap-Pitch Transformation Sequence

Extension:

$$m(\ddot{u} - \ddot{U}_F) - me(\ddot{v}' \cos\theta + \ddot{w}' \sin\theta) - 2m\Omega[\dot{v} - e(\dot{\phi} + \dot{\theta})\sin\theta - e\phi\dot{\theta}\cos\theta]$$

$$- m\Omega^2(x + u - U_F - ev'\cos\theta - ew'\sin\theta) + m\Omega^2\beta_{pc}(w + e\sin\theta$$

$$+ e\phi\cos\theta) - \{EA[u' + k_A{}^2\phi'\,\theta'_{pt} - e_A(v'' + \phi w'')\cos\theta$$

$$+ e_A(\phi v'' - w'')\sin\theta] + E^*A\dot{u}'\}' = A_u \tag{97a}$$

Chordwise bending:

$$m\ddot{v} - me(\ddot{\phi} + \ddot{\theta})\sin\theta - 2m\Omega\beta_{pc}\dot{w} - m\Omega^2(v + e\cos\theta - e\phi\sin\theta)$$

$$- \{me[\Omega^2 x(\cos\theta - \phi\sin\theta) + 2\Omega\dot{v}\cos\theta]\}'$$

$$+ 2m\Omega(\dot{u} - \dot{U}_F - e\dot{v}'\cos\theta - e\dot{w}'\sin\theta) - (Tv')'$$

$$+ \{EAe_A u'(\phi \sin \theta - \cos \theta) - EB_2\phi' \theta'_{pt} \cos \theta - EC_2\phi'' \sin \theta$$

$$+ v''[EI_{\eta\eta}(\sin^2 \theta + \phi \sin 2 \theta) + EI_{\zeta\zeta}(\cos^2 \theta - \phi \sin 2 \theta)]$$

$$+ w''[(EI_{\zeta\zeta} - EI_{\eta\eta})(\sin \theta \cos \theta + \phi \cos 2 \theta)]$$

$$+ E^*(I_{\zeta\zeta}\cos^2 \theta + I_{\eta\eta}\sin^2 \theta)\dot{v}'' + \phi' w' GD_2\}'' = A_v \qquad (97b)$$

Flapwise bending:

$$m\ddot{w} + me(\ddot{\phi} + \ddot{\theta}) \cos \theta + 2m\Omega\beta_{pc}\dot{v} - (Tw')'$$

$$- \{me[\Omega^2 x(\sin \theta + \phi \cos \theta) + 2\Omega\dot{v} \sin \theta] - \phi' v'' GD_2\}'$$

$$+ \{- EAe_A u'(\phi \cos \theta + \sin \theta) + EC_2\phi'' \cos \theta - EB_2\phi'\theta'_{pt} \sin \theta$$

$$+ v''[(EI_{\zeta\zeta} - EI_{\eta\eta})(\sin \theta \cos \theta + \phi \cos 2 \theta)]$$

$$+ w''[EI_{\eta\eta}\cos^2 \theta + EI_{\zeta\zeta}\sin^2 \theta + \phi(EI_{\zeta\zeta} - EI_{\eta\eta})\sin 2 \theta]$$

$$+ E^*(I_{\zeta\zeta}\sin^2 \theta + I_{\eta\eta}\cos^2 \theta)\dot{w}''\}'' = A_w - m\Omega^2\beta_{pc}x \qquad (97c)$$

Torsion:

$$mk_m^2(\ddot{\phi} + \ddot{\theta}) + m\Omega^2\phi(k_{m_2}^2 - k_{m_1}^2) \cos 2 \theta$$

$$+ me[\Omega^2 x(w' \cos \theta - v' \sin \theta) - (\ddot{v} - \Omega^2 v)\sin \theta + \ddot{w} \cos \theta]$$

$$
\begin{aligned}
&+ m\Omega^2 e\phi(v \cos\theta - x\beta_{pc}\sin\theta) \\
&- 2m\Omega[e \sin\theta(\dot{u} - \dot{U}_F) - (k_{m_2}^2 - k_{m_1}^2)\dot{v}' \sin\theta\cos\theta \\
&- \dot{w}'(k_{m_2}^2 \sin^2\theta + k_{m_1}^2 \cos^2\theta) - e\beta_{pc}(\dot{v}' \cos\theta + \dot{w}' \sin\theta)] \\
&+ 2m\Omega e\dot{v}(w' \cos\theta - v' \sin\theta) - me\phi(\ddot{v} \cos\theta + \ddot{w} \sin\theta) \\
&+ \{EC_1\phi'' + EC_2[w'' \cos\theta - v'' \sin\theta - \phi(w'' \sin\theta + v'' \cos\theta)] + E^*C_1\dot{\phi}''\}'' \\
&- \{EAk_A^2 u'(\theta_{pt}' + \phi') + EB_1\,\theta_{pt}'^2\phi' + GJ\phi' + v''w'\, GD_2 \\
&+ EB_2[\theta_{pt}'(\phi v'' - w'') \sin\theta - \phi' v'' \cos\theta - \phi' w'' \sin\theta \\
&- \theta_{pt}'(v'' + \phi w'') \cos\theta] + E^*B_1\theta_{pt}'^2\dot{\phi}' + G^*J\dot{\phi}'\}' \\
&+ (EI_{\zeta\zeta} - EI_{\eta\eta})[v''w'' \cos 2\theta + (w''^2 - v''^2) \sin\theta\cos\theta] \\
&+ EB_2\phi'\theta_{pt}'(v'' \sin\theta - w'' \cos\theta) - EC_2\phi''(v'' \cos\theta + w'' \sin\theta) \\
&+ EAe_A u'(v'' \sin\theta - w'' \cos\theta) = M_\phi - m\Omega^2\beta_{pc} ex \cos\theta \\
&- m\Omega^2(k_{m_2}^2 - k_{m_1}^2)\sin\theta\cos\theta \qquad (97d)
\end{aligned}
$$

The boundary conditons follow from

$$(s_1 + d_1)\delta u \Big|_0^R = 0$$

$$(k_2 - Tv' + s_4' + s_8' + d_4')\delta v \Big|_0^R = 0$$

$$(s_4 + s_8 + d_4)\delta v' \Big|_0^R = 0$$

$$(k_4 + Tw' + s_9 - s_5' - d_5')\delta w \Big|_0^R = 0$$

$$(s_5 + d_5)\delta w' \Big|_0^R = 0$$

$$(s_7 - s_2' + s_3 + d_3 - d_2' + d_6)\delta\phi \Big|_0^R = 0$$

$$(s_2 + d_2)\delta\phi' \Big|_0^R = 0 \qquad (98)$$

The tension T is the same as that given in equation 95 above for the flap-lag-pitch sequence. The generalized aerodynamic forces A_u, A_v, A_w, and A_ϕ are obtained from equations 70 and 71 using equations 89 and 90 in conjunction with equations 66 and 68 and retaining terms through second degree in the dependent variables. Again, for simplicity, comparisons with the literature will be made using U_T, U_P, and $\dot{\epsilon}$.

COMPARISONS AND DISCUSSION

In this section the nonlinear aeroelastic equations of motion developed above will be compared with some of the more recent literature dealing with flexible rotor blades. These comparisons will reveal several differences with the present results. In order to most clearly explain these differences attention will be directed to one or more of several of the fundamental quantities which are needed in the development of the equations of motion including: (1) the rotational transformation matrix relating the angular orientation of the deformed and the undeformed blade; (2) the blade curvature expressions; (3) the strain expressions; and (4) the tangential and perpendicular components of the blade velocity.

Reference 1 derived the nonlinear aeroelastic equations for bending and torsion of a rotating beam. Although no indication was given in this reference as to the sequence in which the rotations were imposed while developing the equations of motion and the resultant rotational transformation matrix was not given, it can be shown that the transformation matrix which leads to the displacement field given in equations B-1 and B-2 of reference 1 is given by

$$[T]_{ref.\ 1} = \begin{bmatrix} 1 & v' & w' \\ -(v'\cos\phi + w'\sin\phi) & \cos\phi & \sin\phi \\ v'\sin\phi - w'\cos\phi & -\sin\phi & \cos\phi \end{bmatrix} \qquad (99)$$

To second degree, equation 99 can be written as

$$[T]_{ref.\ 1} = \begin{bmatrix} 1 & v' & w' \\ -v' - w'\phi & 1 - \frac{\phi^2}{2} & \phi \\ v'\phi - w' & -\phi & 1 - \frac{\phi^2}{2} \end{bmatrix} \qquad (100)$$

Comparing the result given in equation 100 to either equations 11 or 27 after setting the pretwist to zero, it is clear that reference 1 has retained the nonlinear terms involving ϕ but has discarded all second-degree terms which involve squares and products of v' and w'. It is interesting to note that in this case [T] is the same for both of the rotational sequences considered herein. Specializing equation 20 of reference 13 to the case of small strains, the curvatures can be written in terms of the direction cosines relating the deformed and undeformed blade coordinates as

$$\omega_{x_3} = \ell_3\ell_2' + m_3m_2' + n_3n_2'$$

$$\omega_{y_3} = \ell_1\ell_3' + m_1m_3' + n_1n_3'$$

$$\omega_{z_3} = \ell_2\ell_1' + m_2m_1' + n_2n_1' \qquad (101)$$

Substituting the direction cosines from equation 100 into equation 101, the curvatures become

$$\omega_{x_3} = \phi' + v''w'$$

$$\omega_{y_3} = - w'' + v''\phi$$

$$\omega_{z_3} = v'' + w''\phi \qquad (102)$$

Using the orthogonality relations

$$\ell_2\ell_3 + m_2m_3 + n_2n_3 = 0$$

$$\ell_1\ell_3 + m_1m_3 + n_1n_3 = 0$$

$$\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0 \tag{103}$$

the curvature relations given by equations 101 can be written in the alternative form

$$\omega_{x_3} = -(\ell_2\ell_3' + m_2m_3' + n_2n_3')$$

$$\omega_{y_3} = -(\ell_3\ell_1' + m_3m_1' + n_3n_1')$$

$$\omega_{z_3} = -(\ell_1\ell_2' + m_1m_2' + n_1n_2') \tag{104}$$

Equations 104 lead to the curvatures

$$\omega_{x_3} = \phi' - w''v'$$

$$\omega_{y_3} = -w'' + v''\phi$$

$$\omega_{z_3} = v'' + w''\phi \tag{105}$$

Now the curvatures obtained using the relations given in equations 101 and 104 should be identical. However, comparing the resulting expressions given by equations 102 and 105, it is seen that the bending curvatures are in agreement but the torsional curvatures differ in the nonlinear terms. This disagreement is a direct consequence of the use of a rotational transformation matrix (equation 100) which was partially linearized. In fact, as pointed out in reference 13, the second-degree terms in the curvature expressions have no meaning if all the second-degree terms are not retained in the transformation matrix. The fact that the nonlinear terms in the bending curvatures are in agreement is only fortuitous. It is easy to verify that the curvatures obtained from equations 101 and 104 are identical if the complete second-degree expression for [T], such as given by equations 11 or 27, is used. Employing results based on the use of equation 100, reference 1 concluded that the derivatives of the elastic axis strains ($\frac{d\hat{\varepsilon}_{\eta\eta}}{dx}$, $\frac{d\hat{\varepsilon}_{\zeta\zeta}}{dx}$, $\frac{d\hat{\varepsilon}_{\eta\zeta}}{dx}$ in the notation of reference 1) were not small compared to the elastic axis curvatures ($\kappa_{\eta\eta}$, $\kappa_{\zeta\zeta}$, and $\kappa_{\eta\zeta}$, in the notation of that reference). This conclusion is at variance with that of reference 11. In this connection, it should be noted that if the complete second-degree expression for [T] is employed rather than that given in equation 100, the derivatives in question are in fact zero to second degree.

Reference 2 derived the transformation matrix relating the deformed and undeformed blade coordinates for a lag-flap-pitch rotational sequence in connection with efforts aimed at identifying the effects of certain second-order terms associated with combined flapwise and edgewise bending on the expression for the angle of attack (and hence lift) of an elastic rotor blade. While expressing the Euler rotation angles ζ, β, and θ in terms of the dependent variables v, w, and ϕ, reference 2 took exception to the third Euler angle θ by first arbitrarily defining the torsional rotation rate ω_i (ω_{x_3} in the present notation) as being equal to ϕ' in the absence of pretwist and then "solving" a differential equation for [T] to obtain θ.

To second degree in the dependent variables this procedure gives

$$\theta = \phi - \int_0^r v''w'dr \tag{106}$$

The nonlinear term appearing in equation 106 was designated "kinematic pitch rotation" in reference 2. Since the transformation matrix [T] is a function of θ, the substitution of θ as given by equation 106 into equation 3 of reference 2 will lead to a second-degree expression for [T] in which the kinematic pitch rotation term appears explicitly throughout the matrix. Using this expression for [T], reference 2 obtained U_T and U_P and then wrote the expression for the lift, from which the expression for the angle of attack could be identified. The resulting expression for the angle of attack, given in equation 8 of reference 2, contains the kinematic pitch rotation term $-\int_0^r v''w'dr$. In contrast, the present result for [T], given in equation 27, and U_T and U_P, given by equations 89a and 89b, do not contain the kinematic pitch rotation term. Reference 2 obtained this term because it identified ω_i as being equal to ϕ' and regarded θ as an unknown, rather than identifying θ as being equal to ϕ and taking ω_i as an unknown. The reason for proceeding in this manner was not given in reference 2. As already pointed out in references 13 and 14, the identification of ω_i with ϕ' is at variance with the literature dealing with the elasticity of beams in which θ is set equal to ϕ and ω_i as well as the two bending curvatures are regarded as the unknowns. If one proceeds in this latter manner, the solution of equation 4 of reference 2 for ω_i yields

$$\omega_i = \phi' + v''w' \tag{107}$$

and the kinematic pitch rotation will not appear in either the expression for [T] or in the expression for the angle of attack which is in agreement with the present results. Thus, the kinematic pitch rotation term does not exist and its appearance in equations 3 and 8 of reference 2 is spurious.

The development of the nonlinear equations of motion for coupled bending and torsion of a twisted rotor blade in a vacuum was given in reference 4 and forms the basis of the numerical results given earlier in reference 3 in which aerodynamics were included. The equations given in references 3 and 4 were derived using the transformation matrix developed in reference 2 but extended to include pretwist. This extension, which is given in the Appendix of reference 4, follows reference 2 in arbitrarily assuming that the torsional curvature is known and given by

$$\omega_i = \theta'_{pt} + \phi' \tag{108}$$

and then solving for the third Euler angle $\bar{\theta}$. To second degree, this yields

$$\bar{\theta} = \theta_{pt} + \phi - \int_0^x v''w'dx \tag{109}$$

which is equation A-6 in reference 4. Since [T] is a function of $\bar{\theta}$, the substitution of equation A-6 of reference 4 into equation A-3 will lead to a second-degree expression for [T] in which the kinematic pitch rotation term appears. The present result for [T] given in equation 27 does not contain this term. This term was obtained in reference 4 because that reference arbitrarily identified ω_i as being equal to $\theta'_{pt} + \phi'$ and regarded $\bar{\theta}$ as an unknown rather than taking

$$\bar{\theta} = \theta'_{pt} + \phi' \tag{110}$$

and, following customary practice in the elasticity literature, treating ω_i as an unknown just like the bending curvatures. If the latter approach is adopted, one finds that the torsional curvature is given by

$$\omega_i = \theta'_{pt} + \phi' + v''w' \tag{111}$$

and that the kinematic pitch rotation term does not appear in [T], in agreement with the present results. Thus, the kinematic pitch rotation term of reference 4 is also spurious. Using the transformation matrix which includes the kinematic pitch rotation terms, reference 4 derived the components of the strain tensor in Eulerian coordinates, which is based on the use of coordinates for the final or deformed state. The present expressions for the strains given in equations 30 have been derived using a Lagrangian description of the deformation, which is based on the use of coordinates of the initial or undeformed state. This is the usual approach employed in solid mechanics. Comparing the present strains given in equations 30 with those given in equations 24 to 26 of reference 4, one can identify one difference in the expression for the extensional strain and one difference in each of the expressions for the shear strains. The difference in the extensional strain involves the term $\frac{1}{2}(v'^2 + w'^2)$ which appears in equation 24 of reference 4 but does not appear in the present result given in equation 30a. The absence of this term in equation 30a is due to the fact that the present development explicitly includes the second degree terms associated with the foreshortening of the elastic axis due to bending in the axial displacement field (see equation

29) whereas reference 4 does not. However, the foreshortening effect may be accounted for without explicitly including these terms in the axial displacement field, although special considerations are required in this case (reference 24). Reference 4 has apparently followed such as alternative approach. The difference in the shears is reflected in the absence of the term involving $v''w'$ in the expressions for the shear strains given in reference 4. The absence of the term $v''w'$ in the shear strains of reference 4 is related to the use of the incorrect expression for the third Euler angle given by equation 109, rather than the correct one given by equation 110, in the expression for [T]. It is interesting to note that if the correct torsional curvature expression given by equation 111 were used in the general expressions for the shear strains given in equations 20b and 20c of reference 4, the term $v''w'$ would appear in the final shear strains given in equations 25 and 26 of that reference. The differences associated with the term $v''w'$ will lead to non-vanishing differences in the generalized elastic forces appearing in the final equations of motion of the present study and those appearing in reference 4. In particular, when compared to the present results reference 4 lacks the terms $(\phi' w' GD_2)''$ in the chordwise equation of motion, $-(\phi' v'' GD_2)'$ in the flapwise equation of motion, and $(v'' w' GD_2)'$ in the torsion equation of motion. These terms are of the same order as other terms arising from the strain energy which are retained in the final equations of reference 4. If one invokes the basic assumption of reference 4 that terms of $O(\varepsilon^2)$ are negligible compared to unity and applies it to the inertial terms of the present extensional equation given by equation 97a, the resulting equation (with damping set to zero) is in agreement with that given by equation 61a of reference 4. Except for the additional terms mentioned above, the

present chordwise and flapwise bending equations given in equations 97b and 97c (with damping set to zero) are in agreement with the corresponding equations in reference 4. The present torsion equation given in 97d is in agreement with that of reference 4 if the ordering scheme of that reference is applied to the present torsion equation. However, the ordering scheme employed in reference 4 requires that exception be taken to the ordering scheme as applied to the torsional equation in order not to lose terms which, from both physical and mathematical considerations, should be retained. The present torsion equation contains many elastic and inertial terms which reference 4 does not have because the ordering scheme adopted in the present development is more general than that employed in reference 4.

Reference 5 extends the development of reference 4 to include variable structural coupling and hover aerodynamics and presents an analytical trend study of stability in hover. Some earlier numerical results based on these equations for the case of zero structural coupling were given in reference 3. Since reference 5 is based on the dynamic and elastic development given in reference 4, all comments made in the discussion of reference 4 are also applicable to reference 5 and will not be repeated here. Thus, the comments to be made here will be directed only to the aerodynamic aspects of reference 5. Since the present expressions for the lift and moment as a function of U_T, U_P, and $\dot{\varepsilon}$ given in equations 61 and 68 are in agreement with the corresponding expressions given in reference 5, any differences between the present results and those given in reference 5 can be identified most easily by comparing the resultant expressions for U_T, U_P, and $\dot{\varepsilon}$. Specializing the present results given in equations 89 and 90 to the case of hover with zero cyclic pitch and no kinematic coupling and applying the same ordering scheme as in reference 5, the present expressions for U_T, U_P, and $\dot{\varepsilon}$ are given below along with those from reference 5 (in the present notation) for comparison.

Present:

$$U_T = \Omega x + \dot{v}$$

$$U_P = - \Omega x(\theta + \phi + v'w') - \dot{v}(\theta + \phi) - \lambda\Omega R + \dot{w} + \Omega v(\beta_{pc} + w')$$

$$\dot{\varepsilon} = \dot{\phi} + \Omega(\beta_{pc} + w') + \dot{v}'w' \qquad (112)$$

Reference 5:

$$U_T = \Omega x + \dot{v}$$

$$U_P = - \Omega x(\theta + \phi + v'w' - \int_0^x v''w'dx) - \dot{v}(\theta + \phi) - \lambda\Omega R + \dot{w} + \Omega v(\beta_{pc} + w')$$

$$\dot{\varepsilon} = \dot{\phi} + \Omega(\beta_{pc} + w') \qquad (113)$$

Comparing these two sets of expressions one can observe two differences. The first difference is related to the presence of the so-called kinematic pitch rotation term $\int_0^x v''w'dx$ in the expression for U_P of reference 5. This term does not appear in the present result for U_P. As discussed earlier while commenting on references 2 and 4, this term is spurious. The second difference involves the term $\dot{v}'w'$ which does not appear in the expression for $\dot{\varepsilon}$ given in reference 5. This difference is also a consequence of the arbitrary identification of ω_i with ϕ'. A curious aspect of the ordering scheme of reference 5 is its assumption of $\theta = O(1)$ in the structural portion of the derivation but $\theta = O(\varepsilon)$ in the aerodynamic portion. There appears to be no rationale for this dual ordering scheme. In this connection, it is interesting to note that if θ_{pt} is taken to be $O(1)$ in the entire development, several terms must be discarded in the resulting expression for U_p of reference 5 according to the ordering scheme therein, including the spurious kinematic pitch rotation term.

In the comparisons made with the literature thus far, the point has been made that references 2 to 5 have all obtained a spurious nonlinear term as a consequence of an arbitrary identification of the torsional curvature with either $\theta'_{pt} + \phi'$ or ϕ', depending on whether pretwist is present or not. This appears to have a direct bearing on a recent criticism in the literature (reference 25) of some work of Prandtl and Reissner (references 26 and 27) dealing with the lateral buckling of slender cantilever beams. Using equation 5 of reference 2, reference 25 obtained the result given in equation 3.6 therein which can be written in the present notation as

$$\hat{\phi} = \phi - \int_0^x v''w_o'dx \tag{114}$$

where the subscript denotes the pre-buckled condition. The lateral shear force N_1, which follows from a first integral of Kirchoff's lateral equilibrium equation, was then given as

$$N_1 = P\left[\phi - \int_0^x v''w_o'dx\right] \tag{115}$$

where P is the tip load acting in the plane of maximum flexural rigidity.

This result, as pointed out in reference 25, does not agree either with the expression given in reference 26 or with the different expression given in reference 27. If Kirchoff's lateral equilibrium equation is solved for N_1 using the torsional curvature associated with a flap-lag-pitch rotational sequence and the resulting expression is perturbed about the equilibrium position specified in reference 25 one obtains

$$N_1 = P(\phi - w_o' v') \tag{116}$$

which agrees with Prandtl's result as given in reference 25. If one solves for N_1 using the torsional curvature associated with a lag-flap-pitch sequence one obtains

$$N_1 = P\,\phi \tag{117}$$

which is Reissner's result as given in reference 25. This suggests that the differences between the results of Prandtl and Reissner might be associated with their use of torsional curvature expressions corresponding to two different rotational transformation sequences.

Nonlinear aeroelastic equations of motion describing the coupled flap-lag-torsion dynamics of a cantilevered rotor blade in hover were presented in reference 8, which is based on an earlier development given in reference 6. More recently, reference 8 has been extended in reference 10 to include various unsteady aerodynamic theories in a hover stability analysis. Although the majority of the immediate comments will be directed to references 6 and 8, they are also applicable to reference 10. The expression for the extensional strain given in equation 1 of reference 8 was obtained by following the procedure given in reference 28 but retaining second order quantities associated with elastic torsion. This expression is different from the one given in equation 30a herein, but they can be shown to be equivalent by eliminating u' from equation 30a in the manner indicated in reference 28. This is done by making use of the equilibrium condition that the integral of the longitudinal stress over the cross section must be equal to the total tension T, solving the resulting expression for u', and then substituting back into equation 30a. It is interesting to note that the analytical development in reference 6, which is cited as the basis of the inertial and aerodynamic expressions given in reference 8, is based on the use of a rotational transformation matrix [T] in which all the nonlinear terms involving v' and w' have been discarded. The matrix [T] will not be orthogonal to second degree in the dependent variables and, in line with the reasoning given earlier in the discussion of reference 1, the retention of the nonlinear terms involving v'' and w'' in equation 1 of reference 8 is not consistent with the ordering scheme employed for [T] in reference 6. The expressions for the shear strains are not given in reference 8. However, from the form of the generalized elastic forces given therein, it appears that the shear strains used in reference 8 are

$$2\,\varepsilon_{x\eta} = -\,\zeta\phi'$$

$$2\,\varepsilon_{x\zeta} = \eta\phi' \tag{118}$$

the effects of warping being assumed zero. Comparing these expressions to those given herein by equations 30b and 30c after setting u, θ, and λ to zero, it can be seen that reference 8 does not have the nonlinear term involving $v''w'$. It thus appears as though there is a difference in the level of approximation ascribed to the extensional and shear strains. As already indicated earlier, these terms lead to second-degree terms in the equations of motion which are of the same order as the terms which are retained in reference 8. These differences can be identified by comparing the elastic terms in equations 97b, 97c, and 97d of the present results to equations 2 to 4 of reference 8. The expressions for U_T and U_P given in reference 8 were developed in reference 6 and written with respect to the blade local coordinate system resulting after the lag and flap rotations are imposed whereas the present results are written with respect to the blade local coordinate system resulting after the three rotations lag-flap-pitch are imposed. The two sets of expressions can be compared, however, after setting θ, ϕ, and u to zero in the present results. For this comparison, both the present results and those of reference 8 in the present notation are given below.

Present:

$$U_T = \underline{\Omega v v'} + \dot{v} + r\Omega\left(1 - \underline{\frac{v'^2}{2}}\right) - \underline{\Omega U_F} + \underline{\Omega R\lambda\beta_{pc}v' - \Omega\beta_{pc}w}$$

$$U_P = \Omega v w' + \dot{w} - \Omega R\lambda\left(1 - \underline{\frac{w'^2}{2}}\right) - r\Omega v'w' + \underline{\Omega R\lambda\beta_{pc}w'} + \Omega\beta_{pc}v \qquad (119)$$

Reference 8:

$$U_T = \dot{v} + r\Omega$$

$$U_P = \Omega v w' + \dot{w} - \Omega R\lambda + \Omega\beta_{pc}v \qquad (120)$$

If the ordering scheme of reference 8 is adopted, the underlined terms in equation 119 can be discarded. The resulting expression for U_T is then in agreement with that of reference 8 but the resulting expression for U_P is not in agreement with that of reference 8. The present expression for U_P contains the additional term $-r\Omega v'w'$ which is of the same order as the nonlinear term which is retained in reference 8. Reference 8 did not obtain this term because the transformation matrix [T] which was used to obtain the velocity expressions did not contain the nonlinear terms involving v' and w'. It should be noted that the term $-r\Omega v'w'$ "opposes" the term $\Omega vw'$ in the expression for U_P. If this expression is specialized to the case of a rigid articulated blade, these two terms will cancel (reference 14). Consequently, the expression for U_P given in reference 8 will not reduce to the correct expression for U_P for a rigid articulated blade having a lag-flap hinge sequence. The contribution of the blade pitching angular velocity to the lift and moment are reflected in the $\dot{\varepsilon}$ terms in the present development. In reference 8 this effect appears to be reflected solely in the terms $\theta_G^* + \phi^*$ which appear in the component lift expressions. The terms $\Omega\beta_{pc} + \Omega w' + \dot{v}'w'$, which appear in equation 90 of the present results, are not contained in reference 8.

References 7 and 9 considered the case of coupled flap-lag stability in forward flight. The comments will be directed to reference 9 but are equally applicable to reference 7. The present results for U_T and U_P with θ, ϕ, and u set to zero are given below as well as those of reference 9 in the present notation for comparison.

Present:

$$U_T = \dot{v} + r\Omega(1 - \underline{\frac{v'^2}{2}}) + \mu\Omega R \sin\psi\ (1 - \underline{\frac{v'^2}{2}}) + \mu\Omega R \cos\psi\ v'$$

$$\underline{+ \Omega vv' - \Omega U_F + \mu\Omega R\lambda\beta_{pc} v' - \Omega\beta_{pc} w}$$

$$U_P = \dot{w} - \Omega R\lambda(1 - \underline{\frac{w'^2}{2}}) + \Omega vw' + \mu\Omega R \cos\psi\ w' + \underline{\Omega R\lambda\beta_{pc} w'}$$

$$+ \mu\Omega R\beta_{pc} \cos\psi(1 - \underline{\frac{w'^2}{2}}) - \underline{\underline{r\Omega v'w' - \mu\Omega R \sin\psi\ v'w'}} + \Omega\beta_{pc} v \qquad (121)$$

Reference 9:

$$U_T = \dot{v} + r\Omega + \mu\Omega R \sin\psi + \mu\Omega R \cos\psi\ v'$$

$$U_P = \dot{w} - \Omega R\lambda + \Omega vw' + \mu\Omega R \cos\psi\ w' + \mu\Omega R\beta_{pc} \cos\psi + \Omega\beta_{pc} v \qquad (122)$$

If the ordering scheme of reference 9 is adopted, the singly underlined terms in equations 121 can be discarded. The resulting expression for U_T is then in agreement with that of reference 9 but the resulting expression for U_P is not in agreement with that of reference 9. The present expression for U_P contains two additional terms which are doubly underlined in equation 121. These terms are of the same order as the nonlinear term which is retained in reference 9. As stated earlier, the loss of these terms is due to the partial linearization of the matrix [T]. It is also interesting to note that if μ is taken to be of $O(\varepsilon)$, as in reference 6, rather than of $O(1)$, as in reference 9, the ordering scheme of reference 9 would necessitate discarding the term

$-\mu\Omega R \sin\psi\, v'w'$ and would imply that v' was negligible compared to unity. However, the considerations of appendix B clearly show that one must not assume the bending slopes (that is, w' and v') are negligible compared to unity if one is deriving the nonlinear bending equations.

SOME ADDITIONAL COMMENTS ON THE NONLINEAR EQUATIONS OF MOTION

The present equations for a lag-flap-pitch rotational transformation sequence have been compared with several sets of corresponding equations existing in the literature. Several discrepancies with the present results were identified, particularly in the nonlinear terms. It was shown in the literature that the aeroelastic stability of hingeless rotor blades is sensitive to the nonlinear terms in the equations of motion. Hence, the next step is to solve the present nonlinear equations for the lag-flap-pitch sequence including the new terms in order to assess the significance of the discrepancies identified on aeroelastic stability.

The present report has also examined the implications of the assumed rotational transformation sequence between the coordinates of the deformed and undeformed blade on the form of the final second-degree nonlinear equations of motion. The need for considering the rotational transformation sequence arises from the need to specify the position vector of an arbitrary point on the blade in the deformed configuration. This position vector is obtained by performing a sequence of rotations and translations from inertial axes fixed in space to axes fixed at the arbitrary point on the blade in the deformed configuration. In this case, rotation of coordinate axes corresponds to matrix multiplication and translation of coordinate axes corresponds to matrix addition. The sequence in which the individual rotations due to flapwise bending, chordwise bending, and torsion are imposed is of importance here because of the nonlinear nature of the governing equations of motion. As mentioned earlier in the Introduction, the angles of rotation associated with the deformations must be treated as finite and the transformation matrices associated with the individual rotations are not commutative. In the case of a rigid articulated blade, the physical arrangement of the hinges dictates the order in which the component rotations must be imposed while specifying the position vector to an arbitrary point on the blade. However, if the blade is flexible, the order in which the individual rotations are imposed is a prerogative of the analyst. Out of the six possible rotational transformation sequences, the lag-flap-pitch sequence seems to have been preferred by rotor dynamicists. However, no rationale is given for this preferential treatment. In this connection, it should be mentioned that in other disciplines, the possibility of alternative rotational transformation sequences is admitted (see, for example, references 21, 29, and 30).

The present report has considered two of the six possible rotational transformation sequences which may be imposed between the coordinates of the deformed and undeformed blade while developing nonlinear equations of a rotor blade: flap-lag-pitch and lag-flap-pitch. The two sets of equations resulting from the imposition of these two rotational transformation sequences are different in the nonlinear terms. Some comments on the meaning of the existence of two different sets of equations describing one physical system are in order. From a mathematical point of view, one may interpret these two sets of equations as representing two different nonlinear approximations of a given physical system. A physical interpretation is also possible. If the two sets of equations are specialized to the case of a rigid blade having only rigid body flapping, lagging, and pitching degrees of freedom, the resulting sets of equations will still be different. One set of equations will correspond to those of a centrally-hinged, fully-articulated rigid blade having a hinge arrangement which is flap-lag-pitch and the other set of equations will correspond to those of a centrally-hinged, fully-articulated rigid blade having a hinge arrangement which is lag-flap-pitch. In the special case involving only the rigid body flapping and lagging freedoms, the resulting two sets of equations agree with those previously derived in reference 14 for a rigid articulated rotor blade. This suggests that the two sets of equations obtained herein for the flexible blade reflect two possible virtual hinge sequences. In preliminary investigations, flexible hingeless rotor blades are often analyzed using a rigid articulated blade mathematical model through the concept of virtual hinges. In this connection, it should be remarked that the hinge sequence in the mathematical model for the rigid articulated blade used to analyze a hingeless flexible blade should be compatible with the virtual hinge sequence of the physical system.

Reference 14 showed that there are differences in stability for a rigid articulated blade depending on the hinge sequence. The question which then naturally arises is whether or not there are similar differences in stability for the flexible blade depending on the assumed order of the rotational transformation matrix. The two sets of equations derived herein for the flexible blade differ in the nonlinear terms in both the aerodynamic and elastic terms, in contrast to those for the rigid blade (reference 14) where there are differences only in the aerodynamic terms. In the case of a flexible blade, the effect of the differences in the elastic forces on stability may negate the effect of the differences in the aerodynamic forces on stability and thus lead to a single stability boundary as expected from physical considerations. This remains to be established numerically.

The present equations have been developed in such a manner that reduced degree-of-freedom cases of the general equations can be obtained by simply deleting the equations and dependent variables corresponding to the degrees of freedom which are to be suppressed. In fact, degrees of freedom which are not of interest can be suppressed in this manner at any stage in the present development. This expediency is possible because the present development explicitly considers the axial foreshortening of the elastic axis due to bending. Special considerations are required if foreshortening is not considered explicitly (reference 24). The linear equations of motion are obtained by simply discarding all nonlinear terms in the equations of interest. In particular, the linear coupled flap-lag-torsion equations of reference 28 for the case in which the axis of rotation passes through the elastic axis can be obtained as a special case of the present equations of motion. It should be remarked that even these linear equations require consideration of the geometric nonlinear theory of elasticity in order to obtain the linear tension-torsion term $(Tk_A^2\phi')'$ in the torsion equation and the linear tension-bending terms, $(Tw')'$ and $(Tv')'$, in the flapwise and chordwise bending equations, respectively.

CONCLUDING REMARKS

The second-degree nonlinear aeroelastic equations of motion for a flexible, twisted, nonuniform helicopter rotor blade undergoing combined flapwise bending, chordwise bending, torsion, and extension in forward flight have been derived using the extended Hamilton's principle. The equations have their basis in the geometric nonlinear theory of elasticity and are consistent with the small deformation level of approximation in which the elongations and shears (and hence strains) are negligible compared to unity, but with no restrictions on the rotations of the sections. A mathematical ordering scheme which is consistent with the assumption of a slender beam was adopted for the purpose of systematically discarding elastic and dynamic terms which are of higher order in the resultant equations of motion. The generalized aerodynamic forces are left in general second-degree form from which one can obtain the aerodynamic loading to the order appropriate to any case of interest. The influence of the assumed rotational transformation sequence on the form of the resultant equations of motion was examined for two of the six possible transformation sequences: flap-lag-pitch and lag-flap-pitch. The present results were compared to some of the more recent work on rotor dynamics which is available in the literature. These comparisons indicated several discrepancies with the present results, particularly in the nonlinear terms. The reasons for these discrepancies were explained. On the basis of the comparisons and considerations made herein, the principal findings of the present study may be summarized as follows:

(1) The minimum level of approximation within the geometric nonlinear theory of elasticity which is needed to obtain the second-degree nonlinear

equations of motion for a rotating blade is the case of small deformations in which the elongations and shears (and hence the strains) are negligible compared to unity, with no restrictions on the rotations of the sections. In particular, the level of approximation usually employed for elastic stability (buckling) problems in which the strains and the rotations are negligible compared to unity with the stipulation that the strains are smaller than the rotations is inadequate for the case of the rotor blade.

(2) When deriving second-degree nonlinear equations of motion the angles of rotation associated with the displacements must be treated as finite and one must retain all terms through second degree in the dependent variables in the resultant rotational transformation matrix between the coordinates of the deformed and undeformed blade.

(3) Since the angles of rotation associated with the elastic deformations must be treated as finite, the individual rotation matrices are not commutative and the resulting differential equations of motion are different in some nonlinear terms depending on the sequence in which the individual rotations are imposed.

(4) Several discrepancies in some nonlinear equations of motion existing in the literature are identified and shown to be a consequence of a partial linearization of the resultant rotational transformation matrix between the coordinates of the deformed and the undeformed blade or the use of an incorrect expression for the torsional curvature.

(5) The so-called "kinematic pitch rotation" term which has been identified in the rotor dynamics literature is shown to be spurious.

(6) As a by-product of this study, some comments on a recent criticism in the literature of the work of both Prandtl and Reissner pertaining to the

lateral buckling of cantilever beams were made. It appears that Prandtl's results are consistent with the use of the nonlinear torsional curvature expression corresponding to a flap-lag-pitch rotational transformation sequence while Reissner's results are consistent with the use of the nonlinear torsional curvature expression corresponding to a lag-flap-pitch sequence.

APPENDIX A

A COMMENT ON THE SLENDER BEAM APPROXIMATION AND ATTENDANT ORDERING SCHEME

In the case of a long, slender beam, the assumptions that the shear deformation and rotary inertia are negligible (ref. 15) and that terms of the type $\eta^2 v''^2$, $\zeta^2 w''^2$, and $\eta\zeta w''v''$ may be discarded in the strain expressions (ref. 18), are often imposed in the development of the linear bending equations of motion for both rotating and nonrotating beams. These assumptions are based on physical considerations related to the slenderness of the beam in directions normal to its lengthwise axis. There appear to be no comparable considerations for the more general case of the nonlinear coupled flap-lag-axial-torsion equations of motion in a vacuum, such as those used in stability analyses of flexible helicopter rotor blades. With a view toward providing such considerations, this Appendix will first examine the implications of the slender beam approximation as applied to the linear uncoupled equations of motion of a rotating beam in bending, torsion, and extension and introduce a mathematical ordering scheme which is compatible with the assumption of a slender beam. Using the insight gained in these specialized considerations, this Appendix will then examine the implications of the slender beam approximation as applied to the second-degree nonlinear coupled equations of motion of a rotating beam and introduce an ordering scheme which is appropriate to this case. This latter ordering scheme will form the basis of the development of the equations of motion in the main text.

The assumptions attendant with the hypothesis of a slender beam can be systematized by introducing a parameter ε which is taken to be of the same order as the nondimensionalized bending displacements v/R and w/R. The order of the dependent variables appearing in the equations of motion of this report, as well as the pertinent geometric quantities, are summarized in equation (A1) below:

$$\begin{aligned}
u/R &= O(\epsilon^2) & \eta/R &= O(\epsilon) \\
v/R &= O(\epsilon) & \zeta/R &= O(\epsilon) \\
w/R &= O(\epsilon) & \lambda/R^2 &= O(\epsilon^2) \\
\phi &= O(\epsilon) & \lambda_\eta/R &= O(\epsilon) \\
x/R &= O(1) & \lambda_\zeta/R &= O(\epsilon) \\
\beta_{pc} &= O(\epsilon) & \theta_{pt} &= O(1)
\end{aligned} \qquad \text{(A1)}$$

It should be noted that u/R is $O(\epsilon^2)$ rather than $O(\epsilon)$ like v/R and w/R. This was shown in reference 13.

Considerations for Uncoupled Equations

The linear uncoupled equations of motion of a rotating beam in bending, torsion, and extension can be derived following the same procedure used to derive the nonlinear equations of motion in the main body of this report and, in the absence of shear deformation, pretwist, and precone,* are given by

*For simplicity, pretwist and precone are not included in the considerations of this Appendix. However, their inclusion will not change these considerations.

Bending:

$$\overset{O(\varepsilon^3)}{(EI_1 w'')''} - \overset{O(\varepsilon^3)}{(mk_{m_1}^2 \ddot{w}')'} + \overset{O(\varepsilon^3)}{(mk_{m_1}^2 w')'\Omega^2} + \overset{O(\varepsilon)}{m\ddot{w}} - \overset{O(\varepsilon)}{(Tw')'} = 0 \qquad \text{(A2)}^{\dagger}$$

Torsion (with warping):

$$\overset{O(\varepsilon^3)}{(GJ\phi')'} - \overset{O(\varepsilon^3)}{mk_m^2\ddot{\phi}} - \overset{O(\varepsilon^3)}{m\Omega^2 (k_{m_2}^2 - k_{m_1}^2)\phi} + \overset{O(\varepsilon^3)}{(Tk_A^2\phi')'}$$

$$\underline{- \overset{O(\varepsilon^5)}{(EC_1\phi'')''} - \overset{O(\varepsilon^4)}{2me_\lambda\Omega\dot{\phi}'} + (\overset{O(\varepsilon^5)}{mk_\lambda^4\ddot{\phi}'} - \overset{O(\varepsilon^4)}{2me_\lambda\Omega\dot{\phi}} - \overset{O(\varepsilon^5)}{mk_\lambda^4\Omega^2\phi'})'} = 0 \qquad \text{(A3)}$$

Extension:

$$\overset{O(\varepsilon^2)}{m\ddot{u}} - \overset{O(1)}{m\Omega^2 x} - \overset{O(\varepsilon^2)}{m\Omega^2 u} - \overset{O(\varepsilon^2)}{(AEu')'} = 0 \qquad \text{(A4)}$$

where, from equation A4, the tension T is given by

$$AEu' = T = \Omega^2\int_x^R mx\,dx + \text{terms which give nonlinear terms in the bending and torsion equations} \qquad \text{(A5)}$$

The section properties associated with warping which appear in equation A3 are defined by

$$me_\lambda = \iint \rho\lambda\zeta \; d\eta d\zeta \qquad mk_\lambda^4 = \iint \rho\lambda^2 \; d\eta d\zeta \qquad \text{(A6)}$$

Usual mathematical practice when introducing an ordering scheme is to first nondimensionalize the governing equations and then establish the order of nondimensional parameters appearing as coefficients of the

†The equation given is for vertical bending. One could alternatively use the edgewise bending equation.

(nondimensional) dependent variables in the resulting equations. If this rigorous procedure is followed, one must retain all terms arising in the development of the equations of motion before nondimensionalizing and discarding higher order terms. In the case involving the nonlinear coupled flap-lag-axial-torsion equations of motion, this procedure leads to an almost insurmountable amount of algebra. To circumvent this problem to some extent, usual practice in the literature dealing with the dynamics of flexible rotor blades is to introduce an ordering scheme while developing the dimensional equations of motion. The present development will follow this practice here as well as in the main body of this report. Consistent with this expedient, the order of each term appearing in equations (A2) to (A4) as established using the ordering scheme given in equation (A1) is shown above each term.

Consistent with the assumption of a slender beam, the rotary inertia terms in the bending equation can be discarded as being negligible compared to the translational inertia term in the equation. This means that, as far as the generalized inertia forces in the bending equation are concerned, inertia terms of $O(\varepsilon^3)$ can be discarded compared to inertia terms of $O(\varepsilon)$. The elastic term $(EIw'')''$, although of $O(\varepsilon^3)$, cannot be discarded, however, since both physical and mathematical considerations dictate the retention of this term. Thus, in accordance with slender beam theory, one must retain terms up to $O(\varepsilon)$ in the inertia forces and up to $O(\varepsilon^3)$ in the elastic forces in the linear bending equation. Extrapolating to the nonlinear case then, the second-degree nonlinear bending equation would be obtained by retaining terms up to $O(\varepsilon^2)$, in the inertia forces and up to $O(\varepsilon^4)$ in the elastic forces.

In the linear torsion equation the dominant inertial and elastic terms are of $O(\varepsilon^3)$. The dominant inertial terms are of the same order as the rotary inertia terms which were discarded in the bending equation under the assumption of a slender beam. However, from both physical and mathematical considerations, the third-order inertia terms appearing in the torsion equation cannot be discarded. This implies that the ordering scheme associated with the usual slender beam approximation as applied to the bending equation cannot be applied to the torsion equation. In other words, the slenderness of the beam imposes no restrictions on torsion. In the absence of the (underlined) warping terms, the second-degree nonlinear torsion equation would be obtained by retaining terms through $O(\varepsilon^4)$, that is, one order higher than in the linear equation, just as in the case of bending. The highest order linear terms associated with warping are $O(\varepsilon^5)$. Hence, if one wants all the second-degree nonlinear terms associated with warping in the torsion equation one must retain terms through $O(\varepsilon^6)$.

In the extensional equation the dominant inertial term is of $O(1)$. However, in order to obtain a physically meaningful and mathematically complete linear equation, the inertia terms of $O(\varepsilon^2)$ must be retained. It should be noted that if the same ordering considerations which were applied to the inertial terms in the bending equation are applied to the extensional equation, the $O(\varepsilon^2)$ inertia terms would be discarded. Thus, as for torsion, the ordering scheme associated with the usual slender beam approximation as applied to the bending equation cannot be applied to the extensional equation. Hence, as for torsion, the slenderness of the beam imposes no restrictions on the extensional equation. Recalling that u/R is $O(\varepsilon^2)$, the second-degree nonlinear extensional equation would be obtained by retaining terms up to $O(\varepsilon^4)$ in both the elastic and inertia forces.

The ordering scheme associated with the above considerations for both the linear and nonlinear uncoupled euqations of motion are summarized in table A1 below. The order appropriate to the case in which warping is not considered in the torsion equation is indicated in parentheses. The ordering scheme shown in table A1 was obtained based on considerations for a rotating beam. The same conclusions would have been reached if the beam had been assumed to be nonrotating.

TABLE A1.- ORDERING SCHEME FOR UNCOUPLED EQUATIONS

	Linear Equations		Second-Degree Nonlinear Equations	
	Elastic forces	Inertia forces	Elastic forces	Inertia forces
Bending equations	ε^3	ε	ε^4	ε^2
Torsion equation	$\varepsilon^5(\varepsilon^3)$	$\varepsilon^5(\varepsilon^3)$	$\varepsilon^6(\varepsilon^4)$	$\varepsilon^6(\varepsilon^4)$
Extension equation	ε^2	ε^2	ε^4	ε^4

Considerations for Coupled Equations

The ordering scheme discussed above can be extended to the general nonlinear case in which the flapwise and edgewise bending, torsion, and extension are coupled. Since u/R is of $O(\varepsilon^2)$, the highest order term in the second-degree nonlinear coupled torsion equation in the presence of warping would be of $O(\varepsilon^7)$. Since the extensional freedom does not play a predominant role in the coupled flap-lag-axial-torsion stability of helicopter rotor blades, all the nonlinear terms involving the extensional deformation u will be discarded in the equations. Imposing this assumption, one then only has to retain terms through $O(\varepsilon^6)$ in all the equations. Now all the nonlinear terms of $O(\varepsilon^6)$ in the resulting torsion equation are associated with warping. It is believed that these nonlinear terms have a small effect on stability. Hence, from an engineering point of view, these terms can be discarded in the torsion equation. Furthermore, as far as the torsion equation is concerned, it is believed that both the linear and nonlinear inertia terms of order higher than ε^3 in the torsion equation have a small effect on stability and hence, only terms up to $O(\varepsilon^3)$ in the torsional inertia forces will be retained. This means that one need now only retain terms up to $O(\varepsilon^5)$ in the elastic forces and up to $O(\varepsilon^3)$ in the inertia forces in all the equations. Rigid adherence to this ordering, however, leads to terms in the bending equation which are of the same order as those discarded in the bending equation under the slender beam assumption. Hence, to be consistent, only terms up to $O(\varepsilon^4)$ in the elastic forces and terms up to $O(\varepsilon^2)$ in the inertial forces are retained in the nonlinear coupled bending equations. Since the extensional freedom does not play a major role in the coupled flap-lag axial-torsion stability of helicopter rotor blades, only linear terms will be retained in the extensional equation. Based on all these considerations and judgments, the order of the elastic and inertial terms which are retained in the second-degree nonlinear coupled flap-lag-axial-torsion equations of motion in the present development are given in table A2 below.

TABLE A2.- ORDERING SCHEME FOR SECOND-DEGREE NONLINEAR COUPLED EQUATIONS.

	Elastic forces	Inertial forces
Bending equations	ε^4	ε^2
Torsion equation	ε^5	ε^3
Extension equation	ε^3	ε^3

APPENDIX B

EQUATIONS OF MOTION FOR THE CASE OF SMALL DEFORMATIONS AND SMALL ANGLES OF ROTATION

The nonlinear aeroelastic equations of motion of a rotating helicopter rotor blade are derived in the main body of this report consistent with the case of small deformations in which the elongations and shears (and hence the strains) are negligible compared to unity, with no restrictions on the rotations of the sections. This particular level of approximation was identified and discussed in reference 11 and was further discussed in reference 13, where it was called the case of "small deformations I." Reference 13, following reference 11, also discussed a more restrictive nonlinear level of approximation in which the elongations, shears, and rotations are negligible compared to unity but the rotations are assumed larger than the elongations and the shears. This second level of approximation, which was called the case of "small deformations II" in reference 13, is the one usually employed in elastic stability (buckling) problems of nonrotating structures. It is therefore of interest to examine the form of the nonlinear equations of motion of a rotating blade which are obtained using this second level of approximation and, by comparison with the equations obtained by the first level of approximation in the main body of this report, to ascertain the validity of the second level of approximation for the case of a rotating rotor blade. This comparative investigation is motivated by the results of reference 13 which shows that the beam curvatures for the case of small deformations II are linear. This suggests that the use of this level of approximation is not applicable for the derivation of nonlinear equations of motion.

For the case of small deformations II the rotational transformation matrix relating the deformed and the undeformed coordinates is obtained from reference 13 by replacing θ_{pt} by θ and is given by the single relation

$$[T]_{SDII} = \begin{bmatrix} 1 & v' & w' \\ -v'\cos\theta & \cos\theta & \sin\theta \\ -w'\sin\theta & -\phi\sin\theta & +\phi\cos\theta \\ v'\sin\theta & -\sin\theta & \cos\theta \\ -w'\cos\theta & -\phi\cos\theta & -\phi\sin\theta \end{bmatrix} \tag{B1}$$

Note that the resultant rotational transformation matrix given in equation B1 is linear and is thus independent of the order in which the individual rotations are imposed. Hence, the curvatures which follow from equation B1 are linear and independent of the transformation sequence. In particular, the torsional curvature is given by

$$\omega_{x_3} = \theta'_{pt} + \phi' \tag{B2}$$

Using equations B1 and B2 in equation 9, the components of the position vector for a generic point in the cross section of the blade after deformation are given by

$$x_1 = x + u - U_F - \lambda(\theta'_{pt} + \phi') - v'y_o - w'z_o$$

$$y_1 = v + y_o - \phi z_o$$

$$z_1 = w + z_o + \phi y_o \tag{B3}$$

where

$$x_o = x - \lambda \quad {}_{t}$$

$$y_o = \eta \cos\theta - \zeta \sin\theta$$

$$z_o = \eta \sin\theta + \zeta \cos\theta \tag{B4}$$

Substituting equations B3 and B4 into equation 7 and performing the necessary operations, the three strain components of interest assume the form

$$\gamma_{xx} = \varepsilon_{xx} = u' - \lambda\phi'' - v''(\eta\cos\theta - \zeta\sin\theta)$$

$$-w''(\eta\sin\theta - \zeta\cos\theta) - (\eta^2 + \zeta^2)(\phi'\,\theta'_{pt} + \frac{\phi'^2}{2})$$

$$\gamma_{x} = 2\varepsilon_{x\eta} = -\hat{\zeta}\phi'$$

$$\gamma_{x\zeta} = 2\varepsilon_{x\zeta} = \hat{\eta}\phi' \tag{B5}$$

Using equations B3 and B5, the equations of motion corresponding to the case of small deformations II and the ordering scheme given in Appendix A assume the form:

Extension:

$$m(\ddot{u} - \ddot{U}_F) - me(\ddot{v}'\cos\theta + \ddot{w}'\sin\theta) - 2m\Omega[\dot{v} - e(\dot{\phi} + \dot{\theta})\sin\theta - e\phi\dot{\theta}\cos\theta]$$

$$-m\Omega^2(x + u - U_F - ev'\cos\theta - ew'\sin\theta)$$

$$+ m\Omega^2\beta_{pc}(w + e\sin\theta + e\phi\cos\theta)$$

$$-\{EA[u' + k_A^2\phi'\,\theta'_{pt} - e_Av''\cos\theta - e_Aw''\sin\theta] + E^*\dot{u}'\}' = \Lambda_u \tag{B6a}$$

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Chordwise bending:

$$m\ddot{v} - me(\ddot{\phi} + \ddot{\theta}) \sin\theta - 2m\Omega\beta_{pc}\dot{w} - m\Omega^2(v + e\cos\theta - e\phi\sin\theta)$$

$$- \{me[\Omega^2 x\cos\theta + 2\Omega\dot{v}\cos\theta]\}' - (Tv')'$$

$$+ 2m\Omega(\dot{u} - \dot{U}_F - e\dot{v}'\cos\theta - e\dot{w}'\sin\theta)$$

$$+ \{- EAe_A u'\cos\theta - EB_2\phi'\theta'_{pt}\cos\theta - EC_2\phi''\sin\theta$$

$$+ w''(EI_{\zeta\zeta} - EI_{\eta\eta})\sin\theta\cos\theta + v''(EI_{\eta\eta}\sin^2\theta + EI_{\zeta\zeta}\cos^2\theta)$$

$$+ E^*(I_{\zeta\zeta}\cos^2\theta + I_{\eta\eta}\sin^2\theta)\dot{v}''\}'' = A_v \tag{B6b}$$

Flapwise bending:

$$m\ddot{w} + me(\ddot{\phi} + \ddot{\theta})\cos\theta + m\Omega^2\beta_{pc}x + 2m\Omega\beta_{pc}\dot{v}$$

$$- \{me[\Omega^2 x\sin\theta + 2\Omega\dot{v}\sin\theta]\}' - (Tw')'$$

$$+ \{EC_2\phi''\cos\theta - EB_2\phi'\theta'_{pt}\sin\theta - EAe_A u'\sin\theta$$

$$+ w''(EI_{\eta\eta}\cos^2\theta + EI_{\zeta\zeta}\sin^2\theta) + v''(EI_{\zeta\zeta} - EI_{\eta\eta})\sin\theta\cos\theta$$

$$+ E^*(I_{\zeta\zeta}\sin^2\theta + I_{\eta\eta}\cos^2\theta)\dot{w}''\}'' = A_w \tag{B6c}$$

Torsion:

$$mk_m^2(\ddot{\phi} + \ddot{\theta}) - m\Omega^2\phi(k_{m_2}^2\sin^2\theta + k_{m_1}^2\cos^2\theta)$$

$$+ me[\ddot{w}\cos\theta - (\ddot{v} - \Omega^2 v)\sin\theta]$$

$$- 2m\Omega[e\sin\theta(\dot{u} - \dot{U}_F) - (k_{m_2}^2 - k_{m_1}^2)\dot{v}'\sin\theta\cos\theta$$

$$- \dot{w}'(k_{m_2}^2\sin^2\theta + k_{m_1}^2\cos^2\theta) - e\beta_{pc}(\dot{v}\cos\theta + \dot{w}\sin\theta)]$$

$$+ \{EC_1\phi'' + EC_2(w''\cos\theta - v''\sin\theta) + E^*C_1\dot{\phi}''\}''$$

$$- \{EAk_A^2 u'(\theta'_{pt} + \phi') + EB_1\theta'^2_{pt}\phi' + GJ\phi' + E^*B_1\theta'^2_{pt}\dot{\phi}' + G^*J\dot{\phi}'$$

$$-EB_2[w''(\theta'_{pt} + \phi') \sin\theta + v''(\theta'_{pt} + \phi') \cos\theta]\}'$$

$$= M_\phi - m\Omega^2(k^2_{m_2} - k^2_{m_1}) \sin\theta \cos\theta - m\Omega^2\beta_{pc} ex \cos\theta \tag{B6d}$$

The tension T appearing in equations B6 is given to second order by

$$T = \int_x^R m\Big[-(\ddot{u} - \ddot{U}_F) + e(\ddot{v}' \cos\theta + \ddot{w}' \sin\theta) + 2\Omega[\dot{v} - e(\dot{\phi} + \dot{\theta}) \sin\theta]$$

$$+ \Omega^2(x + u - U_F - ev' \cos\theta - ew' \sin\theta)$$

$$- \Omega^2\beta_{pc}(w + e \sin\theta)\Big] dx \tag{B7}$$

The terms U_F and $\ddot{U}_F$ in the expression for T given in equation B7 lead to third-degree nonlinear terms when T is substituted into equations B6 and can be discarded. Also, after substituting for T in these equations only resulting terms which are consistent with the ordering scheme adopted in Appendix A should be retained.

The generalized aerodynamic forces A_u, A_v, and A_ϕ are obtained from equations 70 and 71 using equations 66 and 68 where U_T, U_P, and $\dot{\epsilon}$ are given by

$$U_T = (v' \cos\theta + w' \sin\theta)(\mu\Omega R \cos\psi + \Omega R\lambda\beta_{pc} - \dot{u} + \Omega v)$$

$$+ (\phi \sin\theta - \cos\theta)[\Omega\beta_{pc}w - \dot{v} - \Omega(x + u - U_F) - \mu\Omega R \sin\psi]$$

$$- (\sin\theta + \phi \cos\theta)(\Omega R\lambda - \mu\Omega R\beta_{pc} \cos\psi - \dot{w} - \Omega\beta_{pc}v) \tag{B8a}$$

$$U_P = (w' \cos\theta - v' \sin\theta)(\mu\Omega R \cos\psi + \Omega R\lambda\beta_{pc} - \dot{u} + \Omega v)$$

$$+ (\sin\theta + \phi \cos\theta)[\Omega\beta_{pc}w - \dot{v} - \Omega(x + u - U_F) - \mu\Omega R \sin\psi]$$

$$+ (\phi \sin\theta - \cos\theta)(\Omega R\lambda - \mu\Omega R\beta_{pc} \cos\psi - \dot{w} - \Omega\beta_{pc}v) \tag{B8b}$$

$$\dot{\epsilon} = \Omega(\beta_{pc} + w') + \dot{\phi} + \dot{\theta}_{kc} + \theta_{1c}\Omega \sin\psi - \theta_{1s}\Omega \cos\psi \tag{B8c}$$

Since the resultant rotational transformation matrix between the deformed and undeformed blade coordinates (eq. B1) is linear and thus independent of the order in which the component rotations are imposed, the resulting nonlinear equations of motion are also independent of the order in which the rotations are imposed and assume the unique form given in equations B6 to B8. Comparing these results to those given in the main body of this report for either the flap-lag-pitch sequence or the lag-flap-pitch sequence one can identify several terms, both linear and nonlinear, appearing in the more complete equations given in the main text which are absent in the equations corresponding to the case of small deformations II derived in this appendix. For example, the terms $m\Omega^2\phi(k_{m_2}^2 \cos^2\theta + k_{m_1}^2 \sin^2\theta)$ are missing in the torsion equation B6d. It should be pointed out that these terms, in combination with the terms $-m\Omega^2\phi(k_{m_2}^2 \sin^2\theta + k_{m_1}^2 \cos^2\theta)$ which do appear in the torsion equation, lead to the well-known linear centrifugal pitching moment term $m\Omega^2\phi(k_{m_2}^2 - k_{m_1}^2) \cos 2\theta$. Also missing from the torsion equation given by B6d are the nonlinear bending-torsion structural coupling terms $(EI_{\zeta\zeta} - EI_{\eta\eta})[v''w'' \cos 2\theta + (w''^2 - v''^2)\sin\theta \cos\theta]$. These terms were first discussed in reference 31. The corresponding nonlinear bending-torsion coupling terms in the chordwise and flapwise bending equations are given by $[(EI_{\zeta\zeta} - EI_{\eta\eta})\cdot(\phi w'' \cos 2\theta - \phi v'' \sin 2\theta)]''$ and $[(EI_{\zeta\zeta} - EI_{\eta\eta})(\phi v'' \cos 2\theta + \phi w'' \sin 2\theta)]''$, respectively. These terms are missing from equations B6b and B6c.

Based on these comparisons it can be concluded that the level of approximation usually employed in elastic stability (buckling) problems wherein the elongations, shears, and rotations are negligible compared to unity but the rotations are assumed larger than the elongations and shears is not adequate to develop the nonlinear equations of motion of a rotating helicopter rotor blade.

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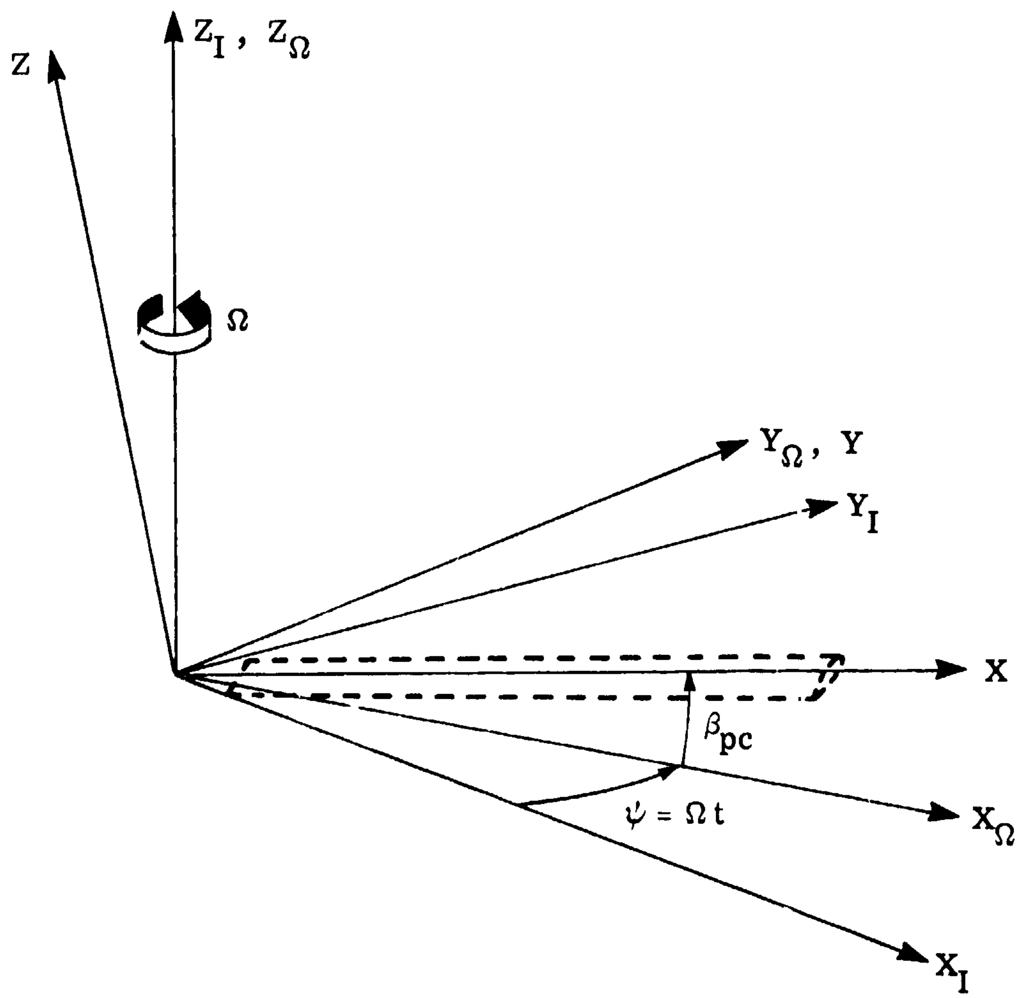

Figure 1.- Coordinate systems of undeformed blade. (Section pitch angle, θ, not shown).

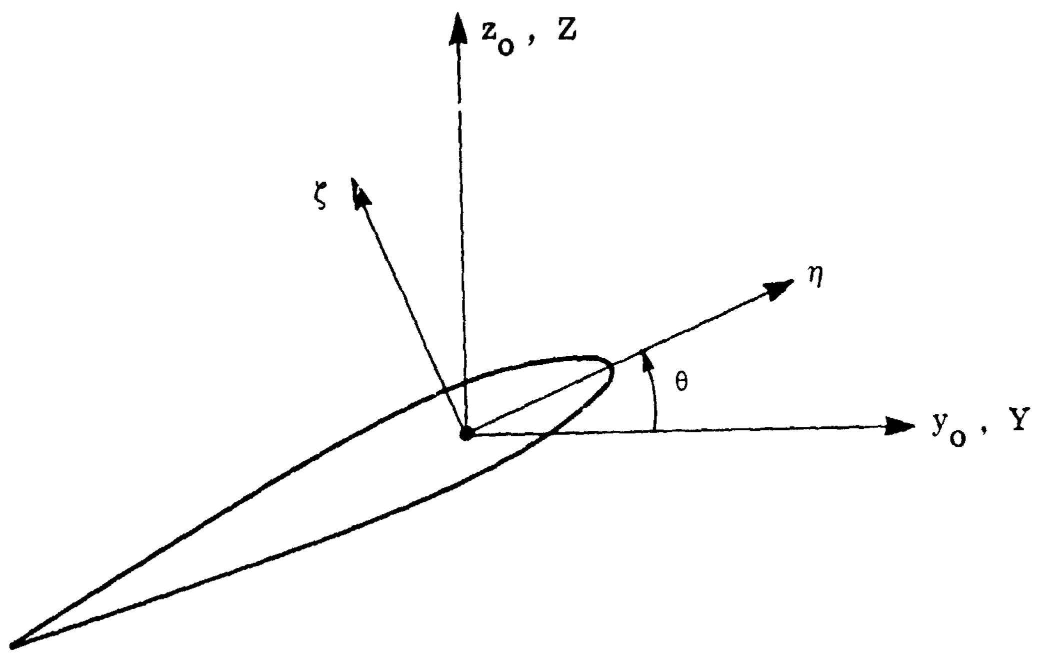

Figure 2.- Coordinate systems of blade cross section.

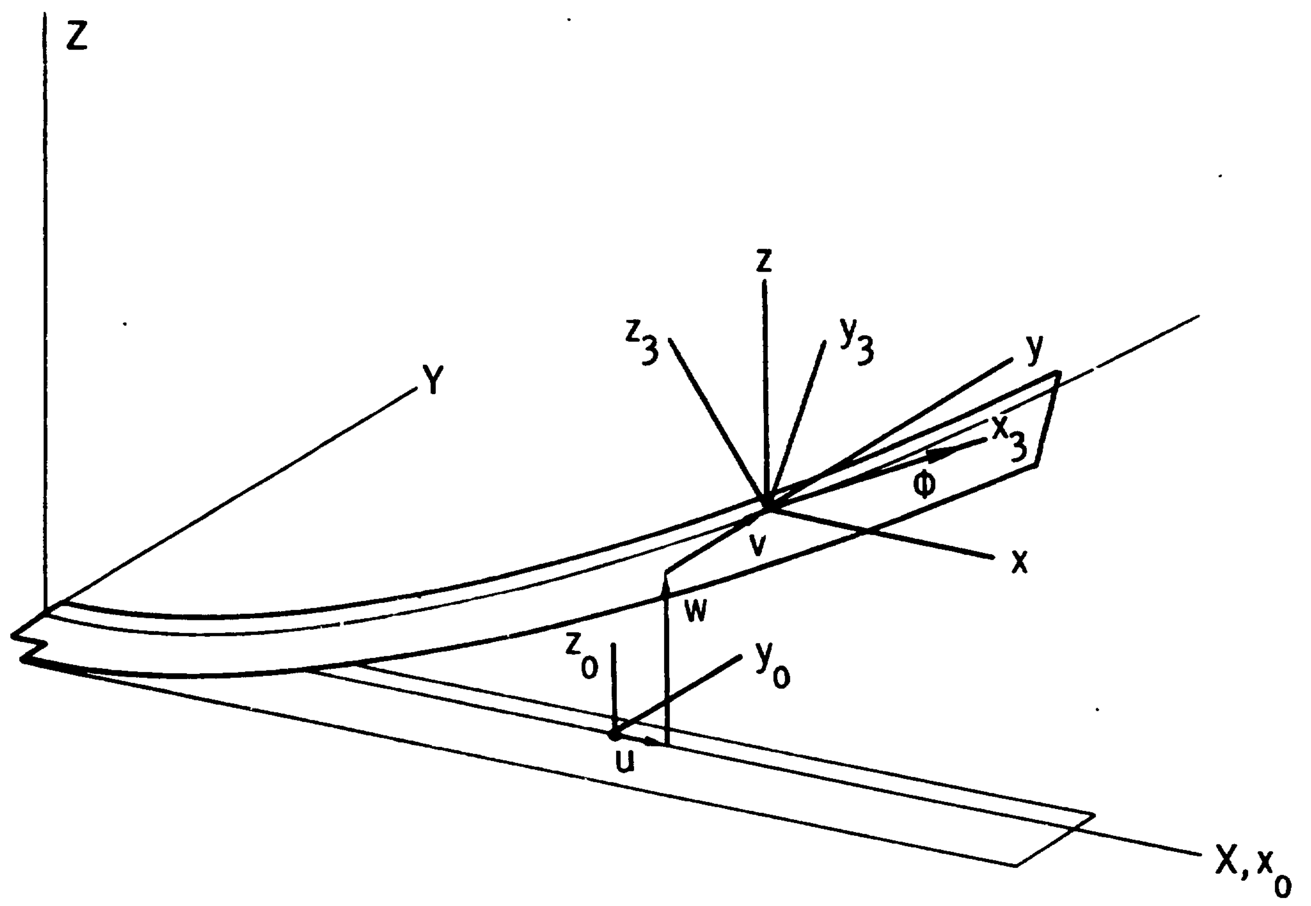

Figure 3.- Schematic representation of undeformed and deformed blade fo. flap-lag-pitch rotational transformacion sequence. (Section pitch angle, θ, not shown).

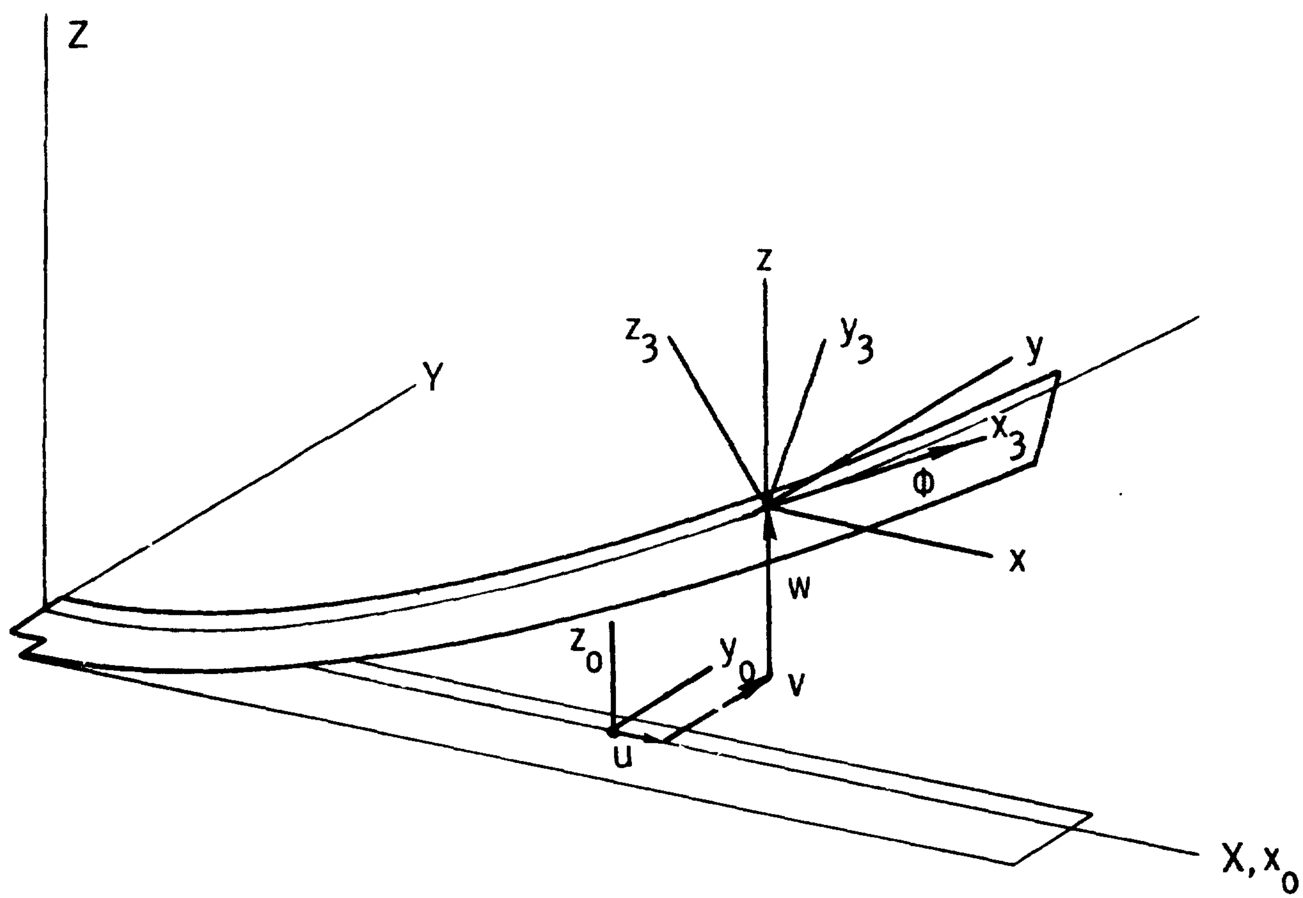

Figure 4.- Schematic representation of undeformed and deformed blade for lag-flap-pitch rotational transformation sequence. (Section pitch angle, θ, not shown).

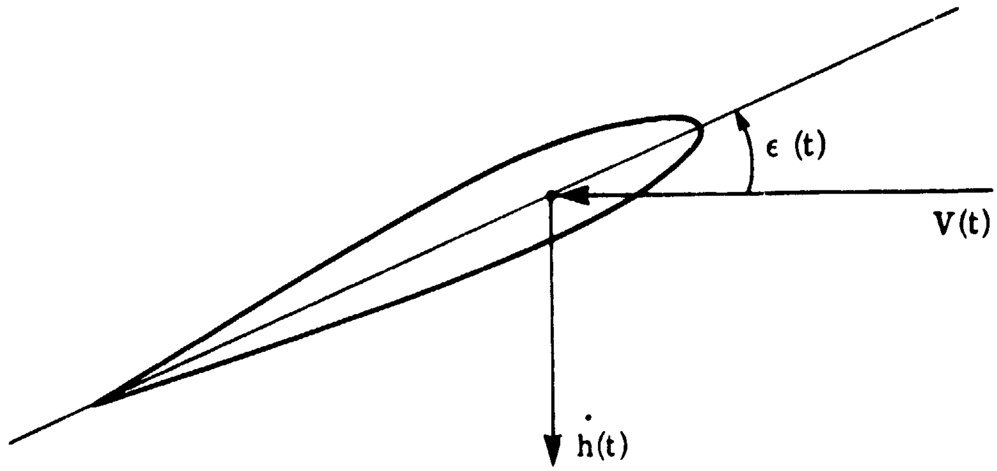

Figure 5.- Cross section of blade in general unsteady motion.

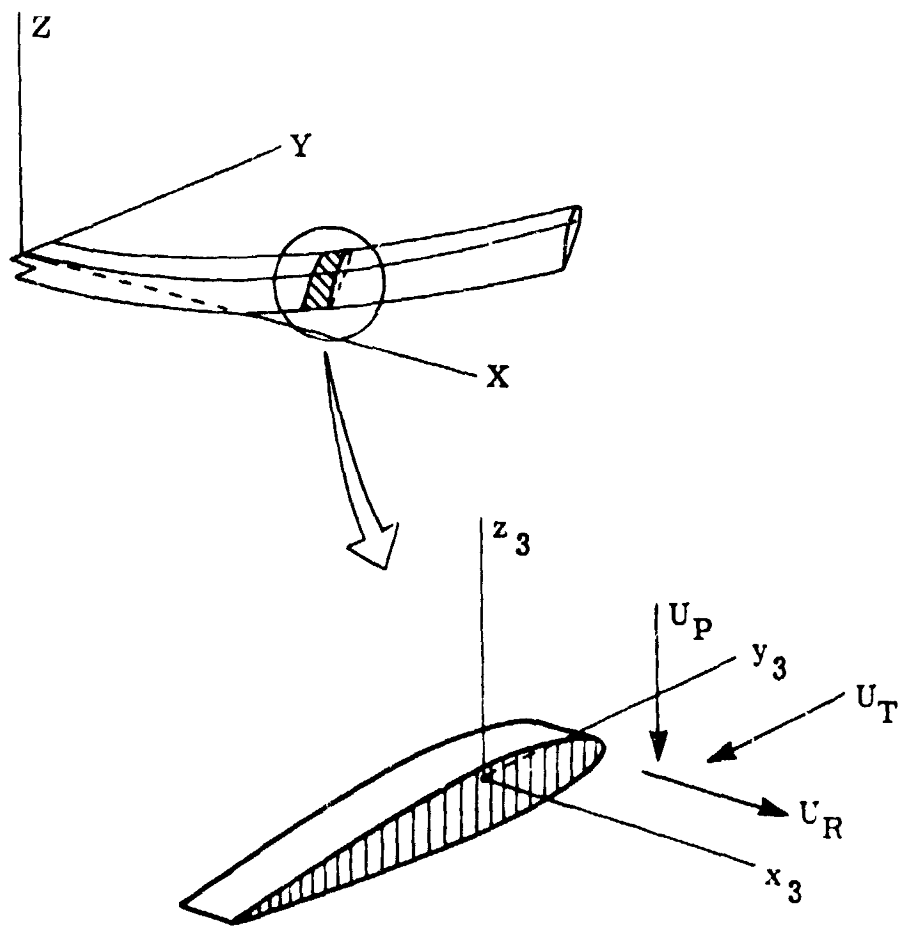

Figure 6.- Relative velocity components at blade cross section.

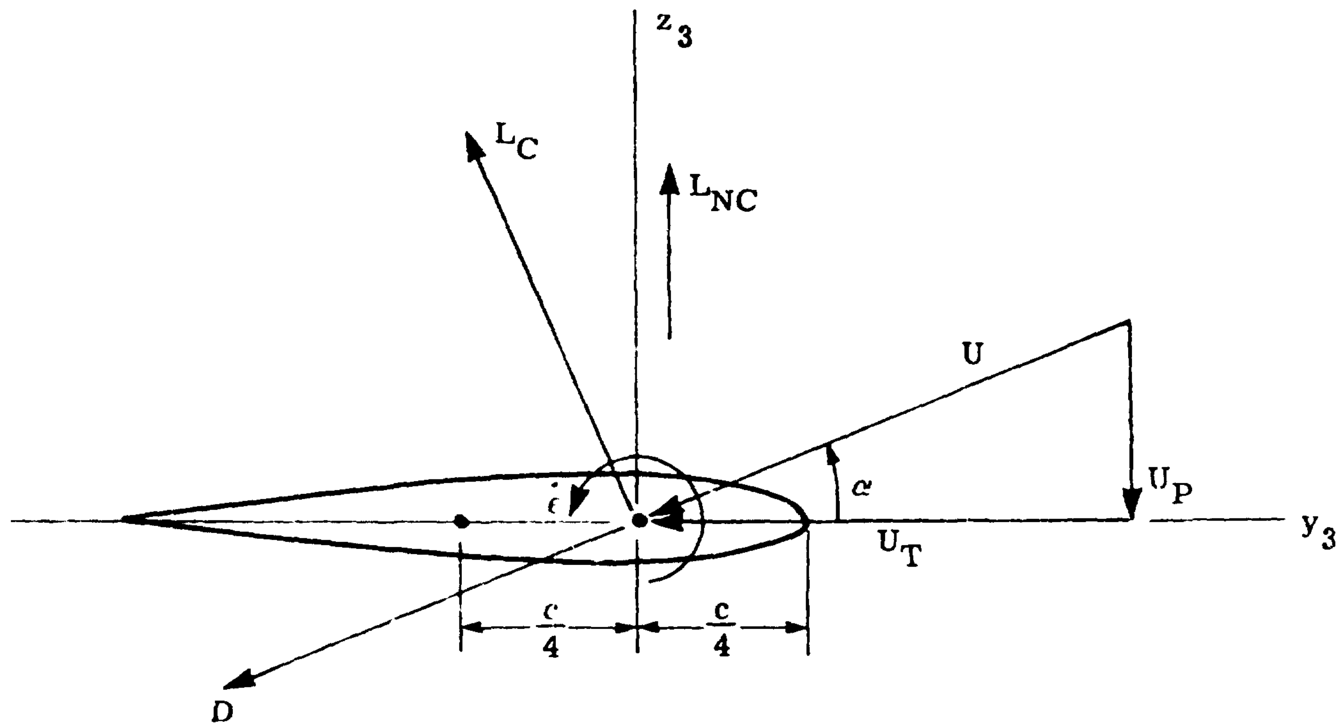

Figure 7.- Blade section inflow geometry and aerodynamic force components.

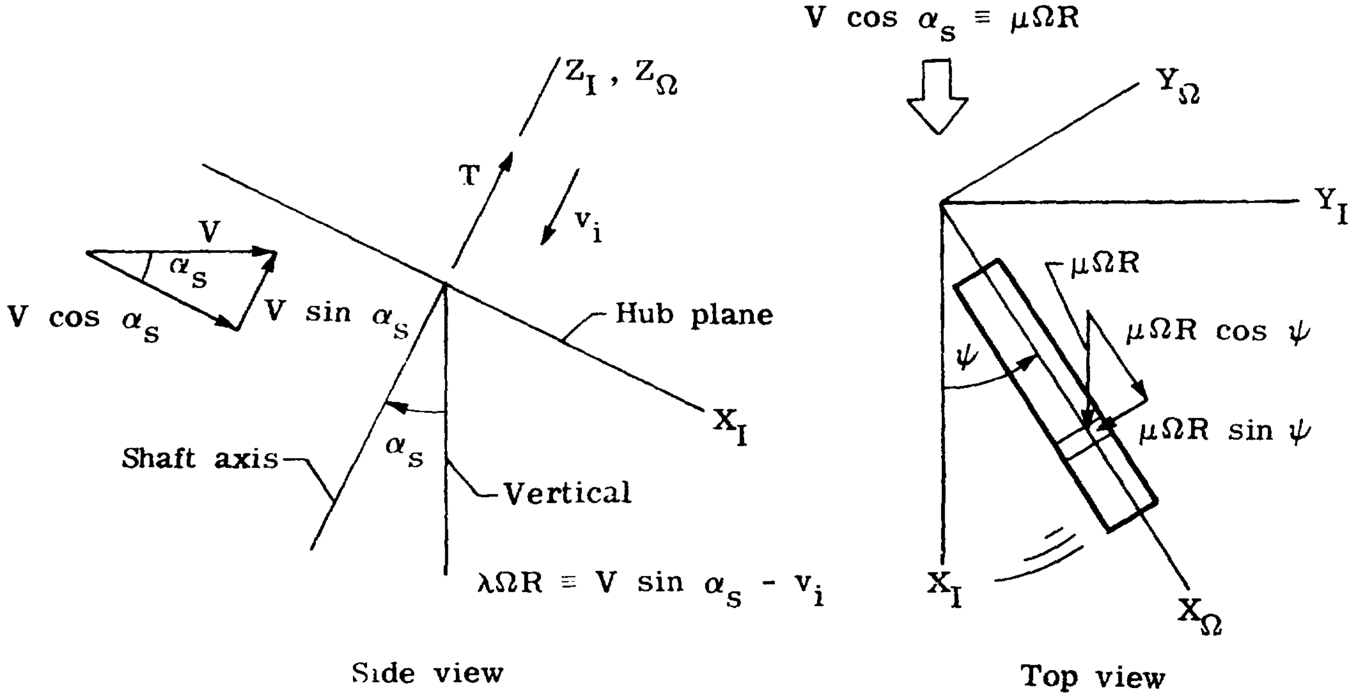

Figure 8.- Geometry for aerodynamic velocity components.

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15. Supplementary Notes K.R.V. Kaza is a Senior Research Associate at the University of Toledo Toledo, OH. This work was initiated while he was a Research Associate at the George Washington University-NASA Joint Institute for Advancement of Flight Sciences, Hampton, VA.

16. Abstract The second-degree nonlinear aeroelastic equations for a flexible, twisted, nonuniform rotor blade which is undergoing combined flapwise bending, chordwise bending, torsion, and extension in forward flight are developed using Hamilton's principle. The equations have their basis in the geometric nonlinear theory of elasticity and are consistent with the small deformation approximation in which the elongations and shears are negligible compared to unity and the square of the derivative of the extensional deformation of the elastic axis is negligible compared to the squares of the bending slopes. No assumption is made regarding the coincidence of the elastic, mass, and tension axes of the blade, although the elastic and aerodynamic center axes are assumed coincident at the blade quarter chord. The blade aerodynamic loading is obtained from strip theory based on a quasi-steady approximation of two-dimensional, incompressible unsteady airfoil theory. The resulting equations are compared with several of those existing in the literature. These comparisons indicate several discrepancies with the present equations, particularly in the nonlinear terms. The reasons for these discrepancies are explained.

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